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IRROTATIONAL FLOW OF FRICTIONLESS FLUIDS, MOSTLY OF INVARIABLE DENSITY

by

Earle H. Kennard

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RESEARCH AND DEVELOPMENT REPORT

February 1967 Report 2299

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FOREWORD

By Dr Frederick H. Todd

This report is a wide-ranging account of the fundamentals of the potential flow of frictionless fluids, and its value is greatly enhanced by the large number of actual examples included in the text. It will be of great value both to the practicing engineer concerned with fluid flows and to the student.

Dr. Earle H. Kennard was formerly Chief Scientist in the Hydromechanics Laboratory at the David Taylor Model Basin, and later head of its Structural Mechanics Laboratory. Throughout his service at the Model Basin he devoted his efforts to the advancement of knowledge in these fields and to the physics of underwater explosions. The value of his work in these areas and in the associated one of structural vibration is well attested by the many papers and reports which he has published.

He has also devoted much time to the education and training of the younger members of the staff. His educational work, indeed, began much earlier as a professor at Cornell University, and unnumbered students have profited from his well-known text book on physics.

His colleagues have learned to respect his judgements, to enjoy his friendship and to appreciate his wit, even though it is sometimes somewhat sharp!

This report is a typical example of Earle Kennard's clear, explanatory writing, combined nevertheless with an admirable economy of words. It is a great pleasure to his many friends and admirers to see it published while Dr. Kennard, though over 80 years of age, is still active and still continuing to work in the field of structural vibration. We look forward to the privilege of making more of his work available to the profession of naval architecture through the medium of Model Basin reports, for it is upon such people as Dr. Kennard and the results of their research that the reputation and image of the establishment depends.

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NOTATION

A, B	Constants
a, b, c	Constants; semi-axes of ellipse or ellipsoid
e	Ellipticity of an ellipse
F	Force on a body
F_1	Force per unit length on a cylinder
f	Constant; sign of a function of a variable
g	A constant
h	A constant, real or sometimes complex
I	Moment of inertia
I'	Equivalent moment of inertia of fluid moving irrotationally around or inside a rotating body
(I')	Symbol denoting that only the imaginary part of the expression following it is to be taken, with omission of the factor i
i	$\sqrt{-1}$
k	A real constant; inertia coefficient
l, m, n	Direction cosines, or constants
M	Mass of a body
M'	Equivalent mass of fluid moving around a body that is in translational motion
M_1, M_1'	Values of M, M' taken per unit length of a cylinder, in cases of two-dimensional motion.
N_1	Torque per unit length on a cylinder
p	Pressure in the fluid
q	Magnitude of particle velocity in the fluid
q_n	Component of the velocity in direction normal to a curve or surface
q_r, q_θ, q_ω	Components of velocity in the directions of polar coordinates r, θ, ω
q_t	Component of velocity in the direction of the tangent to a curve
$q_x, q_{\widetilde{\omega}}, q_\omega$	Components of velocity in the directions of cylindrical coordinates $x, \widetilde{\omega}, \omega$
R	Radius of a circle
(R)	Symbol denoting that only the real part of the expression following it is to be taken
r	A polar coordinate; in two dimensions, distance from a line; in three dimensions, distance from a point
S	Denotes a surface

T	Kinetic energy of the fluid
T_1	In two-dimensional motion, kinetic energy of the fluid between two planes of flow unit distance apart
t	Time; also, an auxiliary complex variable
U	Uniform stream velocity at infinity; velocity of translation of a body
u	Component of velocity in the x-direction
V	Same as U but always specifically parallel to the y-axis
v	Component of velocity in the y-direction
w	In three dimensions, component of velocity in the z-direction; in two dimensions, $w = \phi + i\psi$
X, Y, Z	Components of force on a body
X_1, Y_1, Z_1	Components of force per unit length on a cylinder
x, y	Cartesian coordinates
z	In three dimensions, the third Cartesian coordinate; in two dimensions, $z = x + iy$
α, β, γ	Constants, usually angles
Γ	Circulation around a curve or about a cylinder
ϵ	An angle
ζ	For two dimensions, a complex variable; $\zeta = \lambda - i\mu$ in secs 88-90, $\zeta = \xi + i\eta$ in secs. 81-86 and 106, and $\zeta = -dz/dw$ in secs. 111-117; for three dimensions, a spheroidal coordinate in secs. 137-138
η	An elliptic coordinate in Secs. 81-86, 106
θ	An angle
λ	A special coordinate in Secs. 88-90, 136, 141
λ_1, λ_2	Paraholic coordinates in Sec. 87
μ	Moment of a point dipole, or moment per unit length of a line dipole; a special coordinate in Secs. 88-90, 107, 136, or a spheroidal coordinate in Secs. 137-138
ν	Dipole strength per unit length in Sec. 132; a special coordinate in Secs. 107, 136
ξ	An elliptic coordinate in Secs. 81-86 and 106
ϖ	Distance from a line, sometimes used as a cylindrical coordinate
ρ	Density of the fluid, in dynamical units
ϕ	Velocity potential

ψ	Stream function, for either two-dimensional or axisymmetric three-dimensional motion
ω	Angular velocity, in radians per second; angle about a line, used with x, ϖ as a cylindrical coordinate or with ζ, μ as a spheroidal coordinate
ω_x, ω_y, ω_z	Components of angular velocity about the x-, y-, z-axes

INTRODUCTION

In the work of the David Taylor Model Basin a need was felt some twenty years ago for a collection of the known types of the potential flow of frictionless fluids having a uniform and invariable density. The following report was intended to meet that need in a form convenient for reference.

In Chapter I the chief principles needed in dealing with the potential flow of a frictionless fluid are described. In this chapter, but not elsewhere, variation of the fluid density is sometimes allowed. In Chapter II the use of mathematical complex functions in dealing with two-dimensional problems is explained. Then Chapter III deals with two-dimensional cases and Chapter IV with three-dimensional cases. Sometimes boundary conditions in the form of vortices or vortex lines are allowed. Chapter V lists coefficients of inertia.

The fluid velocity in potential flow is assumed to equal the *negative* gradient of the velocity potential, as in the textbooks of Lamb and of Milne-Thomson. An older assumption was that the velocity equals the *positive* gradient of the potential. The formulas given in this report can be adapted to this older assumption by reversing in all formulas either the potential or all of the fluid velocities wherever these occur.

It was found necessary, however, to limit somewhat the field that is covered. The extensive literature in which incidental use is made of potential flow in treating *practical* flow problems is not even listed. Curved line vortices have not been included, nor interacting spherical boundaries, nor the thin curved stratum that is discussed in Article 80 of Lamb's Hydrodynamics.

This report was finished during the last war, but its great volume was considered to make publication impractical at that time. Publication has finally been effected. No additions have been made, however, to take account of literature published since the war.

CHAPTER I

FUNDAMENTALS OF THE IRROTATIONAL FLOW OF FRICTIONLESS FLUIDS

In this chapter the nature and properties of the irrotational or potential flow of frictionless fluids will be discussed to the extent that is desirable for the understanding and use of the material that forms the body of the report. This chapter may be regarded as an introduction to the subject, but it does not aim at a complete exposition of the mathematical theory of the potential. Further information on the mathematical side may be found in the textbooks of Lamb[1] and Milne-Thomson,[2] in MacMillan's or Kellog's "Theory of the Potential,"[12] or in the periodical literature.

1. PARTICLE VELOCITY AND STREAM LINES

The velocity of the particles in a fluid may vary from point to point in a complicated manner. By a particle of the fluid is meant a portion so small that both its linear dimensions and differences in the motion of its parts may be neglected. The motion of the fluid at any instant can be described completely by specifying the particle velocity at each point.

At any given instant, a set of curves can be drawn such that at every point on a curve its tangent has the direction of the particle velocity at that point. These curves are called streamlines; the aggregate of them is sometimes called a flow pattern. Thus at any given instant the particles are all moving along the streamlines as they exist at that instant.

If the streamlines remain fixed in position, the particles will continue to follow them, and the streamlines will then represent the actual paths of the particles. If the motion undergoes changes, however, the actual paths of the particles may be quite different from any of the instantaneous streamlines. Thus in Figure 1, curves a, b, c may represent streamlines at a time t, and curves a', b', c' may be the streamlines at a later time t', whereas the actual paths pursued by particles P_1, P_2 from time t to t' are as shown by the heavy curves.

An important case in which the paths of the particles coincide permanently with the streamlines is the case of steady motion. The motion of a fluid is called steady when the particle velocity at each point in space remains constant. The velocity of a given particle may vary, however, as it moves from point to point. Motion may be steady when referred to one frame of reference and variable when referred to another. Thus the motion of the air around an airplane in steady flight is a steady motion as seen by the pilot, whereas at a fixed p nt above the ground the velocity of the air changes as the airplane goes past.

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[1]References are listed on page 396.

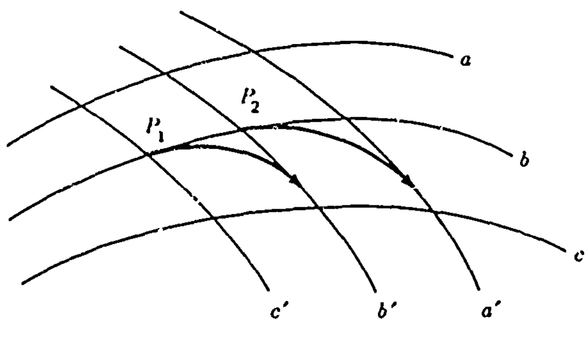

Figure 1 – Two successive positions of several streamlines and, shown by heavy curves, the paths of two particles.

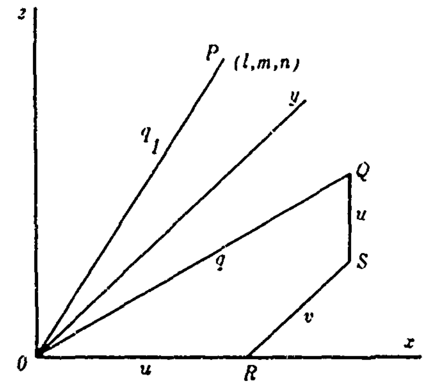

Figure 2 – Relation between the velocity and its components.

In addition to streamlines, the concept of tubes of flow is sometimes useful. A tube of flow is a slender filament of fluid whose bounding surface is composed of streamlines.

The particle velocity is a vector quantity. Its magnitude will be denoted by q; its components in the directions of the x-, y-, z-axes of a rectangular Cartesian coordinate system will be denoted by u, v, w. Thus

$$q^2 = u^2 + v^2 + w^2 \tag{1a}$$

The component q_1 of the velocity in a direction whose direction cosines are l, m, n can then be written

$$q_1 = lu + mv + nw \tag{1b}$$

as is evident from Figure 2, in which the component in the direction OP is represented by the projection on OP of either the vector OQ representing the velocity or of the broken line $ORSQ$, whose segments represent u, v, and w.

Since q, u, v, and w may vary from point to point, and also with the time, they may be regarded as functions of the four variables x, y, z, and t. In steady motion, however, everything is a function of x, y, z only.

2. THE EQUATION OF CONTINUITY

A relation must exist between the motion of a fluid and changes in its density. If, for example, more fluid enters a given volume than leaves it, the density of the fluid in the volume must increase.

Consider a small cubical element of sides δx, δy, δz whose center is situated at the point (x, y, z), as in Figure 3; let it be fixed in size as well as in position in space. Fixing attention first on the increase in mass due to flow through the two faces perpendicular to the

x-axis, the amount of mass per unit time entering the cube through its left-hand face, per unit time, is

$$\left(\rho u - \frac{\partial(\rho u)}{\partial x}\frac{\delta x}{2}\right)\delta y\,\delta z$$

where ρu stands for the value of this quantity at the point (x, y, z). The amount which leaves through the opposite face is

$$\left(\rho u + \frac{\partial(\rho u)}{\partial x}\frac{\delta x}{2}\right)\delta y\,\delta z$$

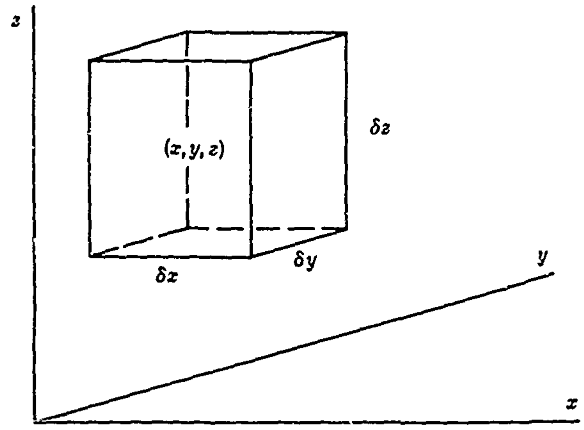

Figure 3 – Illustrating the equation of continuity.

The net increase in mass per unit time due to flow in the x-direction is the inflow minus the outflow or

$$-\frac{\partial(\rho u)}{\partial x}\delta x\,\delta y\,\delta z$$

In an analogous manner the net increases in mass per unit time due to flow in the y- and z-directions are, respectively,

$$-\frac{\partial(\rho v)}{\partial y}\delta x\,\delta y\,\delta z, \quad -\frac{\partial(\rho w)}{\partial z}\delta x\,\delta y\,\delta z$$

The net increase in mass per unit time for all three component directions must equal the increase of mass per unit time within the cube, which is

$$\frac{\partial}{\partial t}(\rho\,\delta x\,\delta y\,\delta z)$$

As the sides of the cube are fixed, this can also be written

$$\frac{\partial\rho}{\partial t}\delta x\,\delta y\,\delta z$$

Collecting the terms and dividing through by the volume or $\delta x\,\delta y\,\delta z$, we obtain the equation of continuity,

$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \qquad [2a]$$

This equation must be satisfied at all points throughout the fluid.

In the subsequent chapters on two- and three-dimensional flow we are mainly concerned with fluids of constant density so that ρ does not vary in space or in time. For this case, $\partial\rho/\partial t$ becomes zero and the equation of continuity takes the form, after canceling out ρ,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad [2b]$$

3. EULER'S EQUATIONS OF MOTION

The equations of motion for a fluid are the mathematical equivalent of Newton's second law of motion, which states that the resultant force on any particle equals the product of the mass of the particle by its acceleration. For convenience we may assume a fluid particle to have the form of a cube whose edges are δx, δy, δz parallel to the x-, y-, z-axes as in Figure 4. Considering the cube as a free body, the forces acting on it may be considered as made up of three parts: compressive or tensile, shear, and external forces, such as gravity. On each face there may be two shear force components parallel to the coordinate axes.

Shear forces, in a fluid, are due to a physical property of the fluid known as viscosity, by virtue of which it offers resistance to motion involving the production of shearing strain. All actual fluids have viscosity, but in some fluids, such as water, the viscosity is quite small. In many flow problems the viscous forces are so small as compared with other forces that their effect may be neglected. This greatly simplifies the mathematical treatment of the problem. Throughout this report, the assumption will be made that the viscosity is zero; this is equivalent to saying that the fluid cannot sustain a shear stress, or that it is frictionless.

Let the pressure or force per unit area at the center of the cube be p, and consider the two faces of the cube normal to the x-axis. Since the pressure will be a function of x, y, z, the average pressure on the left-hand face will be

$$p' = p - \frac{\partial p}{\partial x}\frac{\delta x}{2}$$

and that on the right-hand face,

$$p'' = p + \frac{\partial p}{\partial x}\frac{\delta x}{2}$$

The resultant force due to pressure in the positive x-direction will be the difference between the pressures on these two faces multiplied by the area of a face or

$$-\frac{\partial p}{\partial x}\,\delta x\,\delta y\,\delta z$$

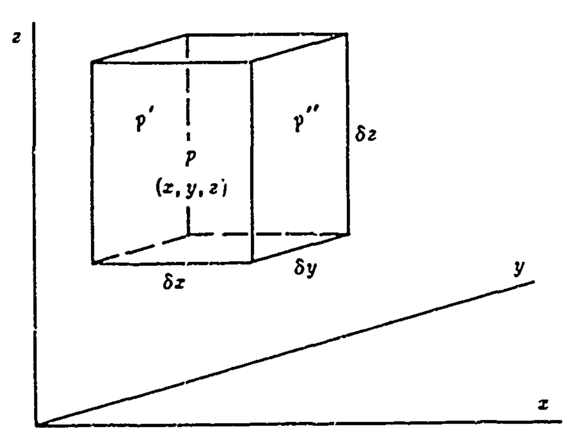

Figure 4 -- Illustrating Euler's equation of motion.

Let X be the component of the external forces per unit mass of fluid in the x-direction. Usually the only source of external forces is gravity. The external force acting on the material in the cube in this direction is then

$$\rho X\,\delta x\,\delta y\,\delta z$$

where ρ is the mass density. Here ρ must be expressed in dynamical units, for example, in slugs per cubic foot or pounds $\sec^2/\text{ft}^4$. As the viscosity has been assumed zero, there can be no other forces acting in the x-direction, and the resultant force on the material in the cube is

$$\left(\rho X - \frac{\partial p}{\partial x}\right)\delta x\ \delta y\ \delta z$$

This force is now equal to the mass of the particle in the cube multiplied by its acceleration in the x-direction, which will be denoted by du/dt. Hence

$$\left(\rho X - \frac{\partial p}{\partial x}\right)\delta x\ \delta y\ \delta z = \rho\ \delta x\ \delta y\ \delta z \frac{du}{dt} \qquad [3a]$$

$$\frac{du}{dt} = X - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

In this equation, u has reference to a certain particle of the fluid, which at a given instant occupies a certain cube but whose position in space varies. Usually, however, it is of no interest actually to follow a particle in its motion; it is more convenient to regard u as a function of position and time or

$$u(x, y, z, t)$$

without regard to the identity of the particle whose coordinates at a particular time t are represented by x, y, z. Viewed from this mathematical standpoint, the change in u during the time dt at the particle just considered may be written

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz + \frac{\partial u}{\partial t} dt$$

Here dx represents the change in the x-coordinate of the particle during the time dt; hence $dx = u\ dt$. Similarly, $dy = v\ dt$, $dz = w\ dt$. Hence, after dividing through by dt,

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial w}{\partial z} \qquad [3b]$$

Here du/dt represents the rate of change of u at a given particle, whereas $\partial u/\partial t$ represents the rate of change in u at a fixed point in space. The last three terms represent an effect due to motion of the particle and are sometimes called convection terms.

Thus, Equation [3a] may be written, together with the analogous equations for the y- and z-directions, in the form known as Euler's equations:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = X - \frac{1}{\rho}\frac{\partial p}{\partial x} \qquad [3c]$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = Y - \frac{1}{\rho}\frac{\partial p}{\partial y} \qquad [3d]$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = Z - \frac{1}{\rho}\frac{\partial p}{\partial z} \qquad [3e]$$

where X, Y, Z are the components of the external force per unit mass. These equations hold whether the density of the fluid is constant or not; in general, p is a function of x, y, z, and t.

For a fluid of constant density, Equations [2b], [3c], [3d], and [3e] constitute four differential equations in four unknowns: u, v, w, and p. As arbitrary constants and functions enter into the solutions of differential equations, boundary conditions are required in order to cube is

$$\left(\rho X - \frac{\partial p}{\partial x}\right)\delta x\ \delta y\ \delta z$$

This force is now equal to the mass of the particle in the cube multiplied by its acceleration in the x-direction, which will be denoted by du/dt. Hence

$$\left(\rho X - \frac{\partial p}{\partial x}\right)\delta x\ \delta y\ \delta z = \rho\ \delta x\ \delta y\ \delta z \frac{du}{dt} \qquad [3a]$$

$$\frac{du}{dt} = X - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

In this equation, u has reference to a certain particle of the fluid, which at a given instant occupies a certain cube but whose position in space varies. Usually, however, it is of no interest actually to follow a particle in its motion; it is more convenient to regard u as a function of position and time or

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$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz + \frac{\partial u}{\partial t}dt$$

Here dx represents the change in the x-coordinate of the particle during the time dt; hence $dx = u\ dt$. Similarly, $dy = v\ dt$, $dz = w\ dt$. Hence, after dividing through by dt,

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial w}{\partial z} \qquad [3b]$$

Here du/dt represents the rate of change of u at a given particle, whereas $\partial u/\partial t$ represents the rate of change in u at a fixed point in space. The last three terms represent an effect due to motion of the particle and are sometimes called convection terms.

Thus, Equation [3a] may be written, together with the analogous equations for the y- and z-directions, in the form known as Euler's equations:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = X - \frac{1}{\rho}\frac{\partial p}{\partial x} \qquad [3c]$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = Y - \frac{1}{\rho}\frac{\partial p}{\partial y} \qquad [3d]$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = Z - \frac{1}{\rho}\frac{\partial p}{\partial z} \qquad [3e]$$

where X, Y, Z are the components of the external force per unit mass. These equations hold whether the density of the fluid is constant or not; in general, p is a function of x, y, z, and t.

For a fluid of constant density, Equations [2b], [3c], [3d], and [3e] constitute four differential equations in four unknowns: u, v, w, and p. As arbitrary constants and functions enter into the solutions of differential equations, boundary conditions are required in order to remain closed, and the circulation about them must remain zero. Since a state of rest is a state in which the circulation vanishes about all closed curves, the application of pressure to the fluid can generate in it only irrotational motion. For this reason, the motion generated by a moving ship in the surrounding water or by an airplane in the air is roughly irrotational, except near the solid surfaces where friction plays a large role.

6. THE VELOCITY POTENTIAL FOR IRROTATIONAL FLOW

Only irrotational motion will be considered hereafter. Its mathematical treatment can be greatly simplified by introducing the *velocity potential.*

If the motion is irrotational within a singly connected region, the integral

$$\int_{P_1}^{P_2} q_s \, ds$$

taken along any path lying in the region and connecting two given points, P_1 and P_2, depends only on the positions of the points P_1 and P_2. The integral is defined as in the last section; the chosen direction of motion along the path is from P_1 toward P_2.

To prove the statement just made, note that any two paths B_1 and B_2, as illustrated in Figure 6a, when taken together form a closed curve around which the circulation vanishes, so that

$$\int_{P_1}^{P_2} (B_1) \; q_s \, d_s + \int_{P_2}^{P_1} (B_2) \; q_s \, d_s = 0$$

But

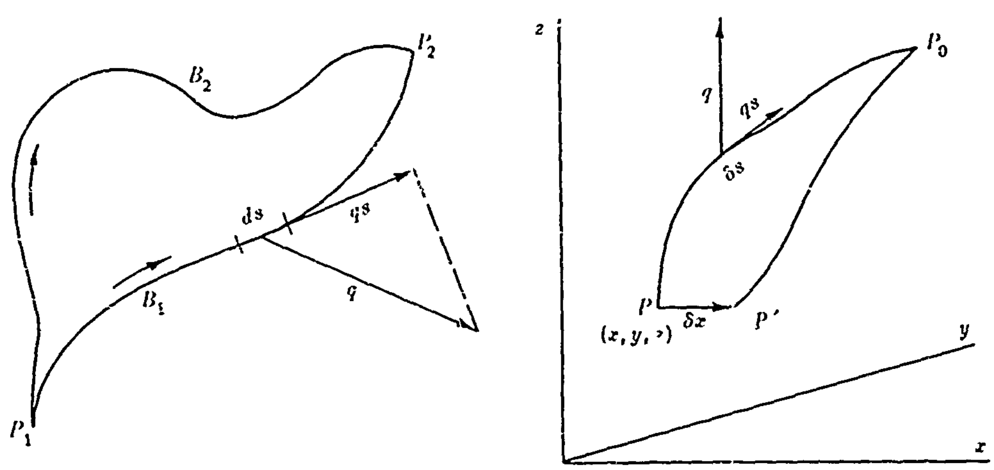


Figure 6a

Figure 6b

Figure 6 – Illustrating properties of the velocity potential.

eliminate them and so make the solution determinate.

4. BOUNDARY CONDITIONS

At the boundaries of the fluid the continuity equation is replaced by special surface conditions. For example, at a fixed boundary it is necessary that the fluid velocity have no component normal to the surface. If l, m, n are the direction cosines of the normal to the surface, this condition requires that

$$lu + mv + nw = 0 \qquad [4a]$$

at every point on the surface; compare Equation [1b].

If the boundary is in motion, the normal component of the fluid velocity at the surface, or $q_n = lu + mv + nw$, must equal the velocity of the surface normal to itself. This is equivalent to saying that the velocity of the fluid relative to the surface is wholly tangential, or that a particle on the surface remains on the surface. A method of finding l, m, n when the equation of the surface is given is derived in Sec. 135.

5. ROTATIONAL MOTION; THE CIRCULATION

In the kinematics of rigid bodies, a distinction is made between translational and rotational motion. In rotational motion, all particles not on the axis go round the axis in circles. An analogous but more general conception of rotational motion in a fluid can be developed as follows.

Consider any closed curve C drawn in the fluid, and choose a positive direction of motion around the curve, as in Figure 5a. At each point of the curve, divide the particle velocity into a component perpendicular to the curve and a component q_s in the direction of the tangent to the curve; let q_s be taken positive when it has the same direction as the chosen positive direction of motion around the curve and negative when it has the opposite direction. In Figure 5a, q_s is positive at P but negative at Q. Multiply each element of length ds along

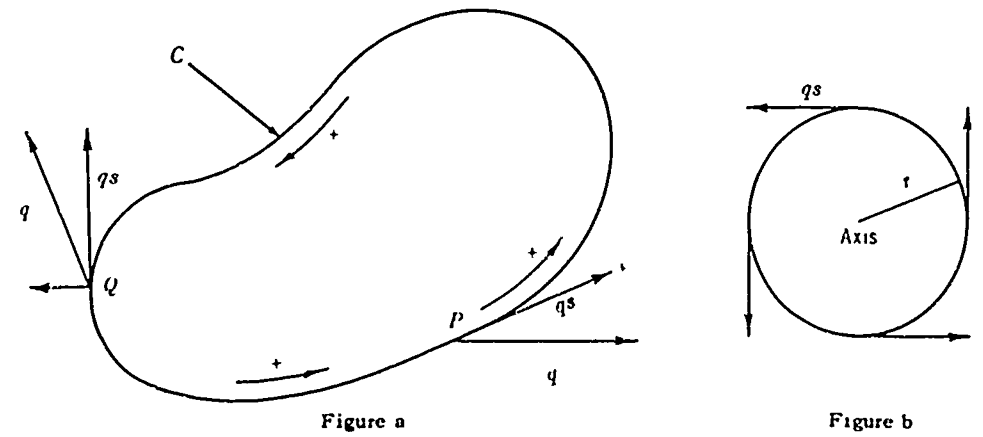


Figure 5 – Illustrating the definition of the circulation.

the curve by the value of q_s at that element and add the products, thus forming the line integral

$$\oint q_s \, ds$$

where $\oint$ indicates that the integration extends around the whole curve. This integral is called the *circulation* of the fluid around the curve C. It may be regarded as a measure of the extent to which the fluid is moving in a rotational manner along this particular path.

The integral can also be written $\oint q_s \, ds = \oint (u \, dx + v \, dy + w \, dz)$ since the direction cosines of ds are dx/ds, dy/ds, dz/ds and hence, by Equation [1b],

$$q_s = u \frac{dx}{ds} + v \frac{dy}{ds} + w \frac{dz}{ds}$$

If the fluid is actually moving like a rotating rigid body, the circulation about any closed curve lying in a plane perpendicular to the axis of rotation is equal to twice the angular velocity of rotation ω multiplied by the area enclosed by the curve. This is easily seen in the special case of a circle whose axis is the axis of rotation, as in Figure 5b; here $q = q_s = \omega r$, where r denotes distance from the axis, and the circulation around the circle is $\oint q_s \, ds = \oint \omega r \, ds = \omega r \oint ds = \omega r \, (2\pi r) = 2(\omega)(\pi r^2)$. The same result is obtained, by evaluation of the integral, for a circle centered anywhere.

Motion of a fluid in which the circulation is zero around any continuously collapsible curve is called *irrotational motion* or *flow*. The significance of the collapsibility of the curve can be illustrated in the space occupied by the body of a doughnut. Any closed curve in this space that is not linked with the central hole can be shrunk down continuously onto a point, or can be deformed in continuous fashion into any other curve of the same type. Curves that link with the hole, on the other hand, although continuously deformable into each other, can never be shrunk below a certain minimum size. A region in which closed curves fall into two classes with respect to collapsibility is called doubly connected; a region in which all curves are completely collapsible is called singly connected. In some cases there are more than two such classes of curves. Regions in which there are at least two classes are called multiply connected. In irrotational motion in a multiply connected region, the circulation is required to vanish only about the closed curves that are continuously collapsible down to a point.

The great importance of irrotational flow arises from the following dynamical theorem, which is proved in Sec. 33 of Lamb's Hydrodynamics.[1]

Suppose that the fluid is frictionless, and that its density, if not uniform and constant, is at least a definite, fixed function of the pressure. Let the external forces be conservative, as are those due to gravity; that is, the total work done by these forces on a given mass vanishes when the mass is carried around any closed curve. Then *the circulation around any closed curve that is allowed to move with the fluid is constant in time.*

It follows from this theorem that, if a mass of frictionless fluid acted on only by conservative forces happens to be moving irrotationally at any instant, it will continue to move irrotationally thereafter. Closed curves moving with the fluid may change their shape, but they remain closed, and the circulation about them must remain zero. Since a state of rest is a state in which the circulation vanishes about all closed curves, the application of pressure to the fluid can generate in it only irrotational motion. For this reason, the motion generated by a moving ship in the surrounding water or by an airplane in the air is roughly irrotational, except near the solid surfaces where friction plays a large role.

6. THE VELOCITY POTENTIAL FOR IRROTATIONAL FLOW

Only irrotational motion will be considered hereafter. Its mathematical treatment can be greatly simplified by introducing the *velocity potential.*

If the motion is irrotational within a singly connected region, the integral

$$\int_{P_1}^{P_2} q_s \, ds$$

taken along any path lying in the region and connecting two given points, P_1 and P_2, depends only on the positions of the points P_1 and P_2. The integral is defined as in the last section; the chosen direction of motion along the path is from P_1 toward P_2.

To prove the statement just made, note that any two paths B_1 and B_2, as illustrated in Figure 6a, when taken together form a closed curve around which the circulation vanishes, so that

$$\int_{P_1}^{P_2} (B_1) \; q_s \, d_s + \int_{P_2}^{P_1} (B_2) \; q_s \, d_s = 0$$

But

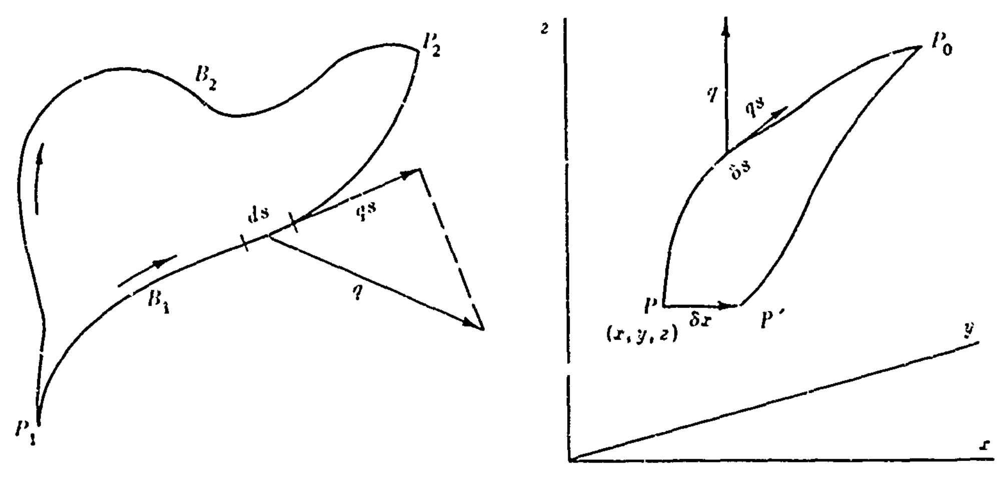


Figure 6 – Illustrating properties of the velocity potential.

$$\int_{P_2}^{P_1} (B_2)\ q_s\, d_s = -\int_{P_1}^{P_2} (B_2)\ q_s d_s$$

since interchange of the limits reverses the direction of motion along the path and hence the positive direction for q_s. Hence

$$\int_{P_1}^{P_2} (B_1)\ q_s\, ds = \int_{P_1}^{P_2} (B_2)\ q_s\, d_s$$

It follows that, if ϕ is defined as

$$\phi = \int_{P}^{P_0} q_s\, d_s \qquad [6a]$$

along any path joining any point P to a fixed point P_0, ϕ will be a single-valued function of the coordinates x, y, z of P.

Furthermore,

$$u = -\frac{\partial\phi}{\partial x}, \quad v = -\frac{\partial\phi}{\partial y}, \quad w = -\frac{\partial\phi}{\partial z}, \quad q^2 = \left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2 \qquad [6b, c, d, e]$$

where u, v, w are the components of the particle velocity at P. For, let the path be drawn so as to run from P straight to a neighboring point P', which is displaced a distance δx from P toward $+x$ without change of y or z, as in Figure 6b. Then, if $\delta\phi$ is the difference in the values of ϕ at P' and at P,

$$\delta\phi = -\int_{P}^{P'} q_s\, ds$$

since the path from P' differs from that drawn from P only by the omission of the additional stretch PP'. But along this stretch $q_s = u$, and is constant in the limit as P' approaches P. Hence

$$\delta\phi = -u\int_{P}^{P'} ds = -u\,\delta x$$

Equation [6b] follows; and Equations [6c, d] can be similarly obtained.

The function ϕ thus introduced is called the velocity potential. If it is known at all points, the particle velocity can be found from it by differentiation. The sign has been chosen in such a way that the potential decreases in the direction of the particle velocity. The relation between the velocity and the velocity potential is the same as the relation between the electric intensity and the electrostatic potential.

Since the position of P_0 is arbitrary, the velocity potential, like the electrostatic potential, contains an arbitrary additive constant. A surface over which ϕ has a constant value is called an equipotential surface. As the derivative of ϕ with respect to any element of distance along the surface is zero, there can be no component of the particle velocity tangent to an equipotential surface; compare Equation [6f] below. Thus the direction of the velocity is everywhere normal to the equipotential surfaces; and, since the streamlines have everywhere the direction of the velocity, the streamlines cut the equipotential surfaces perpendicularly.

Certain other relations between the velocity and the potential may be noted. The component of the velocity in any given direction can be written

$$q_s = -\frac{\partial \phi}{\partial s} \qquad [6f]$$

here $\partial\phi/\partial s$ is the space derivative of ϕ in the given direction or

$$\frac{\partial \phi}{\partial s} = \lim_{\Delta s \to 0} \frac{\Delta \phi}{\Delta s}$$

where Δs is a displacement in that direction and $\Delta\phi$ is the corresponding change in ϕ. The proof is similar to that of Equations [6b, c, d]. By integrating Equation [6f] it is seen that the change in ϕ from one end to the other along any path is

$$\Delta\phi = -\int q_s \, ds \qquad [6g]$$

where the integral is taken along the path.

It is often convenient to use spherical polar coordinates r, θ, ω. Here r is the distance from a fixed origin 0, θ is the angle between the line 0r and a fixed line or axis through 0, and ω is the angle between the plane containing θ and a fixed plane drawn through the fixed axis. The definition is illustrated in Figure 7. A set of Cartesian coordinates is also shown having

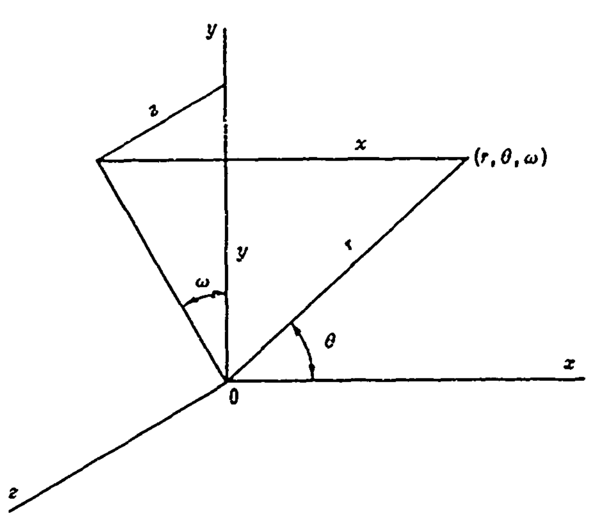


Figure 7 – Illustrating polar coordinates.

the same origin, the fixed plane as the xy-plane and the fixed line as the x-axis. It is evident that

$$x = r\cos\theta, \quad y = r\sin\theta\cos\omega, \quad z = r\sin\theta\sin\omega \qquad [6h, i, j]$$

Any point P can be displaced in such a direction that only one of the coordinates r, θ, ω varies. The three mutually perpendicular directions thus defined may be called coordinate directions; the corresponding elements of distance ds are dr, $r d\theta$ along a circle through Ox, and $r \sin\theta d\omega$ along a circle of radius $r\sin\theta$ and in a plane normal to Ox. From [6f] the components of the velocity in these three directions are

$$q_r = -\frac{\partial\phi}{\partial r}, \quad q_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta}, \quad q_\omega = -\frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\omega} \qquad [6k, l, m]$$

In other cases, cylindrical coordinates x, $\widetilde{\omega}$, ω are useful. Here $\widetilde{\omega}$ denotes distance from the x-axis and ω denotes angular distance around this axis measured from a fixed plane drawn through it; see Figure 8. If the Cartesian axe, are drawn so that ω is measured from the xy plane,

$$y = \widetilde{\omega}\cos\omega, \quad z = \widetilde{\omega}\sin\omega \qquad [6n, o]$$

The elements of distance in the coordinate directions are now dx, $d\widetilde{\omega}$ and $\widetilde{\omega}d\omega$, and the components of the particle velocity in these directions are

$$q_x = -\frac{\partial\phi}{\partial x}, \quad q_{\widetilde{\omega}} = -\frac{\partial\phi}{\partial\widetilde{\omega}}, \quad q_\omega = -\frac{1}{\widetilde{\omega}}\frac{\partial\phi}{\partial\omega} \qquad [6p, q, r]$$

In all of these equations connecting the velocity potential with the velocity, the sign is that of recent textbooks on hydrodynamics. An older usage must be noted in which ϕ is defined as $\phi = \int_{P_0}^{P} q_s\, ds$. Then all differences between values of ϕ are reversed in sign and the signs in equations equivalent to [6b, c, d], [6f, g], [6k, l, m], and [6p, q, r] are all positive.

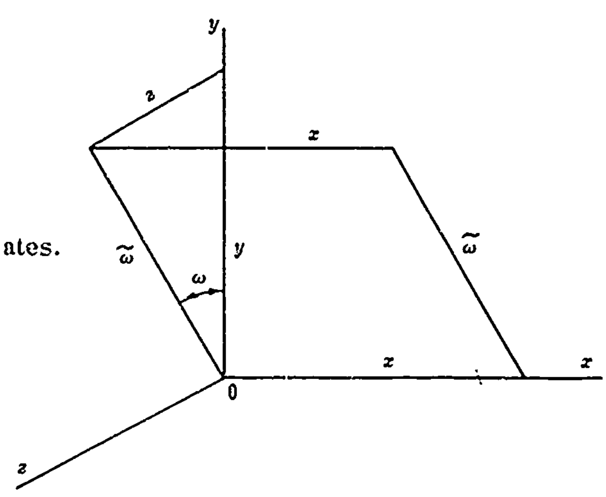


Figure 8 – Illustrating cylindrical coordinates.

7. THE LAPLACE EQUATION

In an incompressible fluid of uniform density the velocity potential satisfies a very simple differential equation. If the velocity potential ϕ is substituted from Equations [6b, c, d] into the equation of continuity, Equation [2b], there results

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \qquad [7a]$$

This equation must be satisfied at every point throughout the region in which irrotational flow exists. It is known as the Laplace equation and is often written in the symbolic form $\nabla^2 \phi = 0$ where ∇^2 stands for the differential operator

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

The Laplace equation is encountered in many other branches of physics, such as electricity, heat flow, and elasticity, and the properties of its solutions are well known.[11, 12, 16] Because of the linearity of the Laplace Equation, its solutions possess the following useful properties. If ϕ is a solution, so is $C\phi$ where C is any constant. If ϕ_1 and ϕ_2 are two solutions, $\phi_1 + \phi_2$ is another solution; the particle velocity corresponding to $\phi_1 + \phi_2$ at any point is the vector sum of the velocities corresponding to ϕ_1 and to ϕ_2. These statements may be verified by substitution in the equation. Finally, if ϕ_1 is a solution, so is $\phi_2 = \partial\phi_1/\partial x$ or $\partial\phi_1/\partial y$ or $\partial\phi_1/\partial z$; for, after substituting ϕ_1 for ϕ in Equation [7a], differentiating with respect to x, for example, and changing the order of integration,

$$\frac{\partial^2}{\partial x^2}\frac{\partial \phi_1}{\partial x} + \frac{\partial^2}{\partial y^2}\frac{\partial \phi_1}{\partial x} + \frac{\partial^2}{\partial z^2}\frac{\partial \phi_1}{\partial x} = 0$$

Thus $\partial\phi_1/\partial x$ is another solution of the Laplace equation.

The problem of determining the motion of a frictionless, incompressible fluid under given conditions thus reduces to the problem of solving the Laplace equation subject to certain boundary conditions. Any solution ϕ of the equation represents a possible type of irrotational flow in which the components of the velocity are given in terms of ϕ by Equations [6b, c, d]. Since the density of the fluid does not occur either in the Laplace equation or in Equation [4a] expressing the usual boundary condition, each type of flow can exist in a fluid of any density. So long as the motion is not too rapid, gases as well as liquids can be assumed to move approximately as if they were incompressible.

Once the velocity at each point is known, the distribution of pressure may be found from the pressure equation to be obtained presently.

8. SOME PROPERTIES OF IRROTATIONAL FLOW

The following properties of the irrotational flow of incompressible fluids may be noted. More rigorous proofs of some of them may be found in Milne-Thomson's Theoretical Hydrodynamics[2] or elsewhere.[12]

a. The distance between two given equipotential surfaces corresponding to slightly different values of ϕ varies in inverse ratio to the magnitude of the velocity q.

For, $\delta\phi = -q\,\delta s$; if $\delta\phi$ is constant, $\delta s \propto \frac{1}{q}$

b. The streamlines are concave toward the side on which the magnitude of the velocity q is larger.

For on the concave side of a streamline neighboring equipotential surfaces, being perpendicular to the line, must converge, as at P in contrast to Q in Figure 9a; hence, by (a), q is greater.

c. The velocity q increases in the direction in which the streamlines converge, and hence is greater near the concave side of an equipotential surface than near its convex side.

For, if the streamlines converge in a certain direction, such as RS in Figure 9b, the associated tube of flow diminishes in cross section in that direction; but the same volume of an incompressible fluid must flow across every cross section of a tube; hence q increases in this direction.

d. In any given region, the maximum velocity must occur at a point on the boundary. The same is true of the minimum velocity unless it is actually zero.

For, suppose a maximum value of q occurred in the midst of the fluid, as at T in Figure 9c. Then q would decrease in all directions from this point. But then the tube of flow containing this point would have to flare in both directions from T by (c), and would also have to be concave inward over the sides of the tube, by (b), which is impossible. The proof for a nonzero minimum is similar.

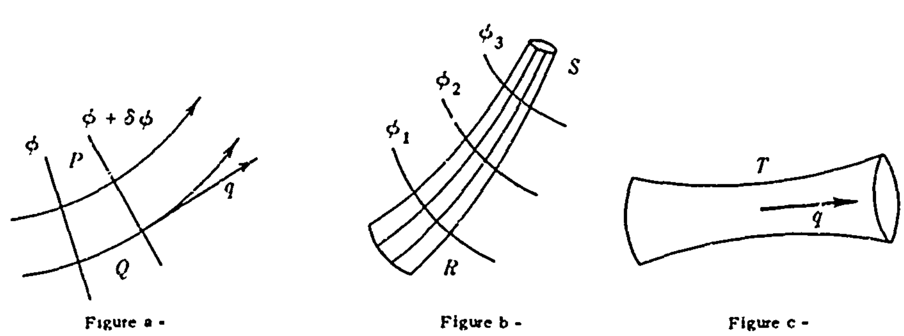


Figure 9 – Illustrating some geometrical properties of streamlines and equipotential surfaces.

e. Within a single connected region the streamlines must begin and end on the boundary. Hence, if the boundary is entirely stationary, so is the fluid.

For, otherwise, closed streamlines would necessarily occur, and this is impossible under the conditions assumed; since, in going around a closed streamline in the direction of the flow, the velocity potential would decrease continually, and hence it could not return to its initial value upon returning to the starting point. Furthermore, no streamline can end on a stationary boundary, since there the normal component of the velocity must vanish.

f. The flow within any region is uniquely determined if the velocity is given, in magnitude and in direction, at all points on the boundary of the region.

For, if two different distributions of velocity satisfying the given conditions were possible, a third one would also be possible in which the velocity is the vector difference of the velocities in the given distributions. In this latter distribution the velocity would be zero at all points on the boundary; hence, by (d), the velocity must vanish everywhere. It follows that the two original distributions of velocity must be identical.

g. In a singly connected finite region, the flow is uniquely determined if the normal component of the velocity is assigned at all points of the boundary.

The proof is similar to that of (f). The difference between two types of flow satisfying the given condition would be another in which the normal component of the velocity vanishes over the boundary, so that no streamlines could begin or end there; hence, as explained under (e), there can be no streamlines, and the two assumed distributions of velocity must be identical.

In all cases, the word boundary may refer either to a physical boundary or merely to a geometric surface drawn through the fluid. Furthermore, except where the contrary is specified, the boundary may lie partly or wholly at infinity.

It may also be noted that differentiation of Equations [6b, c, d] leads to the equations

$$\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} = 0, \quad \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} = 0, \quad \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0. \qquad [8a, b, c]$$

These differential equations may be regarded as an alternative characterization of irrotational motion; for it can be shown by means of a theorem known as Stokes' theorem that the circulation vanishes around any closed curve drawn in a singly connected region in which Equations [8a, b, c] are satisfied. The three left-hand members of the equations are the components of a vector, which is called in vector analysis the curl of the particle velocity and in hydrodynamics is often called the vorticity.

From Equations [8a, b, c] it can be shown, furthermore, that in irrotational flow the motion of any particle is compounded of a motion of translation and one of pure strain. In a pure straining motion there are three mutually perpendicular lines through any particle of the material which do not change their directions; these lines are the strain axes. In rigid-rotational motion, on the other hand, only one line through a given particle retains its direction, namely, the line parallel to the axis of rotation.

9. THE PRESSURE EQUATION FOR IRROTATIONAL FLOW

Let the external forces per unit mass be given by

$$X = -\frac{\partial \Omega}{\partial x}, \quad Y = -\frac{\partial \Omega}{\partial y}, \quad Z = -\frac{\partial \Omega}{\partial z} \qquad [9a, b, c]$$

in terms of a potential function Ω, as is true for forces due to gravity; and let the fluid have the property that its pressure is a definite function of its density. Then, if the motion is irrotational, the equations of motion for a frictionless fluid can be integrated.

On these assumptions, Equation [3c] can be written, after replacing u by $-\partial\phi/\partial x$ in the first term,

$$-\frac{\partial^2 \phi}{\partial t\,\partial x} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial \Omega}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

or, by means of Equations [8b] and [8c], after changing signs,

$$\frac{\partial}{\partial x}\frac{\partial \phi}{\partial t} - u\frac{\partial u}{\partial x} - v\frac{\partial v}{\partial x} - w\frac{\partial w}{\partial x} - \frac{\partial \Omega}{\partial x} = \frac{1}{\rho}\frac{\partial p}{\partial x}$$

Equations [3d] and [3e] become similarly

$$\frac{\partial}{\partial y}\frac{\partial \phi}{\partial t} - u\frac{\partial u}{\partial y} - v\frac{\partial v}{\partial y} - w\frac{\partial w}{\partial y} - \frac{\partial \Omega}{\partial y} = \frac{1}{\rho}\frac{\partial p}{\partial y}$$

and

$$\frac{\partial}{\partial z}\frac{\partial \phi}{\partial t} - u\frac{\partial u}{\partial z} - v\frac{\partial v}{\partial z} - w\frac{\partial w}{\partial z} - \frac{\partial \Omega}{dz} = \frac{1}{\rho}\frac{\partial p}{\partial z}$$

Let these last three equations be multiplied through by dx, dy, dz and added; and let the time t be held constant. Then the first terms give

$$dx\frac{\partial}{\partial x}\frac{\partial \phi}{\partial t} + dy\frac{\partial}{\partial y}\frac{\partial \phi}{\partial t} + dz\frac{\partial}{\partial z}\frac{\partial \phi}{\partial t} = d\left(\frac{\partial \phi}{\partial t}\right)$$

In general, $d\dfrac{\partial \phi}{\partial t}$ would contain also a term $dt\dfrac{\partial}{\partial t}\left(\dfrac{\partial \phi}{\partial t}\right)$; but here it is assumed that $dt = 0$. The next terms in the equations give in the sum

$$u\frac{\partial u}{\partial x}dx + u\frac{\partial u}{\partial y}dy + u\frac{\partial u}{\partial z}dz = \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right)dx \,\ldots\ldots = d\left(\frac{1}{2}u^2\right)$$

The remaining terms give similar results. Thus the final result is

$$d\left(\frac{\partial \phi}{\partial t}\right) - d\left(\frac{u^2}{2} + \frac{v^2}{2} + \frac{w^2}{2}\right) - d\,\Omega = \frac{dp}{\rho}$$

This can be integrated to give

$$\int\frac{dp}{\rho} = \frac{\partial \phi}{\partial t} - \frac{1}{2}q^2 - \Omega + F(t) \qquad [9d]$$

where $q^2 = u^2 + v^2 + w^2$

Here $\int dp/\rho$ can be evaluated when the law of variation of p with ρ is known. The integration "constant" $F(t)$ must be regarded as an arbitrary function of t, for the mathematical reason that the integration involved only x, y, and z, and also for the physical reason that ϕ itself contains an arbitrary "constant" which may be supposed to vary with the time. The presence of this arbitrary term in ϕ limits the usefulness of Equation [9d] in the general case.

If the density is uniform and constant,

$$\int \frac{dp}{\rho} = \frac{p}{\rho} + \text{constant}$$

and

$$\frac{p}{\rho} = \frac{\partial \phi}{\partial t} - \frac{1}{2} q^2 - \Omega + F(t), \qquad [9e]$$

where the arbitrary constant in the integral has been absorbed in $F(t)$. In this case the pressure itself contains an arbitrary additive constant: for it is well known that a uniform pressure p_0 applied to the boundary of an enclosed mass of incompressible liquid merely raises the pressure throughout by the amount p_0 without affecting the motion. To fix the pressure completely, therefore, its value on the boundary must be known. This value then fixes $F(t)$ after the arbitrary additive function of t that occurs in ϕ has been chosen. The equation thus obtained is very useful.

If the only external forces acting are those due to gravity, it is sometimes convenient to simplify the last equation by considering the pressure to be made up of two parts, $p = p_s + p_d$ where p_s is the hydrostatic pressure that would exist if there were no motion and p_d is the dynamic pressure due to changes in velocity. When there is no motion, Equation [9e] gives, with p_s substituted for p and $F(t)$ assumed to be constant,

$$p_s + \rho\Omega - \text{constant}. \qquad [9f]$$

If, then, p is replaced by $p_s + p_d$ in Equation [9e] and p_s then substituted from Equation [9f], the result is

$$\frac{p_d}{\rho} = \frac{\partial \phi}{\partial t} - \frac{1}{2} q^2 + F(t) \qquad [9g]$$

where the constant in Equation [9f] has been absorbed in $F(t)$. In Equation [9g], p is sometimes written for p_d.

When the only external forces are gravitational, if the z-axis is drawn vertically upward, $X = Y = 0$, $Z = -g$, where g is the acceleration due to gravity. Hence Ω is a function of z only, and integration of [9c] gives

$$\Omega = gz + \text{constant}.$$

Dynamical units are to be understood in Equations [9e] and [9g]. The pressure p or p_d may be in pounds per square foot, ρ in slugs per cubic foot, q in feet per second, and t in seconds. The potential ϕ would then be in feet squared per second, since the dimensions of ϕ are those of length times velocity or $L^2 T^{-1}$.

10. THE BERNOULLI EQUATION FOR STEADY IRROTATIONAL FLOW

If the motion is steady, the constant of integration in ϕ can be chosen so that $\partial\phi/\partial t = 0$. Then $F(t)$ reduces to a constant, and the pressure equation [9d] can be written

$$\int \frac{dp}{\rho} + \frac{1}{2}q^2 + \Omega = C \qquad [10a]$$

where C is a constant. Similarly, if ρ is uniform and constant and if the pressure at infinity or on the boundary does not vary with time, Equations [9e] and [9g] become, respectively,

$$\frac{p}{\rho} + \frac{1}{2}q^2 + \Omega = C, \qquad [10b]$$

$$\frac{p_d}{\rho} + \frac{1}{2}q^2 + \Omega = C. \qquad [10c]$$

The value of C, which is not necessarily the same in these three equations, may be found from the known values of the other quantities at some one point, such as a point on the boundary.

In many problems, the motion at infinity is one of uniform flow and $\Omega = 0$. Then, if U is the particle velocity and p_∞ the pressure at infinity,

$$C = \frac{p_\infty}{\rho} + \frac{1}{2}U^2$$

and from Equation [10b], for incompressible fluid,

$$p - p_\infty = \frac{1}{2}\rho(U^2 - q^2) \qquad [10d]$$

It is important to note that the pressure difference, $p - p_\infty$, at any point depends only upon the relative motion between the fluid at the point and the fluid at infinity; in particular, it remains the same if a frame of reference is substituted relative to which U is zero. It is physically obvious that $p - p_\infty$ cannot be altered by a mere change of the frame of reference; and it is easily verified that the resulting changes in U and q are such that the difference $U^2 - q^2$ remains unchanged.

These are various forms of what is commonly called the Bernoulli equation for irrotational motion. For any type of steady flow, whether irrotational or not, equations identical in form can be obtained for any one streamline, but in general the constant may vary from one streamline to another. In irrotational flow the constant C has the same value for all streamlines.

The Bernoulli equation holds throughout any region, large or small, throughout which the motion happens to be irrotational. The region may even surround one or more cylinders about which there is circulation; irrotationality in the neighborhood of each point of the region is all that is required.

11. THE PRESSURE EQUATION FOR ROTATING BOUNDARIES

The following special case may be noted for reference. Suppose that incompressible fluid is set into irrotational motion by the steady rotation of a solid boundary, internal or external, the fluid being otherwise unbounded. Then the flow pattern will obviously be always the same relative to the boundary but at any point fixed in space variations will occur. The distribution of values of the potential ϕ can be imagined to rotate with the boundary, but otherwise to remain unchanged. Since the motion is not steady in space, the Bernoulli equation cannot be used. Let the density ρ be uniform.

Let θ denote an angle of position about the axis of rotation and let the angular velocity of rotation be ω. Then that value ϕ_1 of ϕ which, at time t, is at a point P is carried forward by the rotation during an interval dt to a point P' at which θ is greater by $d\theta = \omega\, dt$. At time t, on the other hand, the value of ϕ at P' was

$$\phi = \phi_1 + \frac{\partial \phi}{\partial \theta} d\theta = \phi_1 + \omega \frac{\partial \phi}{\partial \theta} dt.$$

Thus, during dt, ϕ changes at P' by

$$d\phi = \phi_1 - \left(\phi_1 + \omega \frac{\partial \phi}{\partial \theta} dt\right) = -\,\omega \frac{\partial \phi}{\partial \theta} dt$$

Hence, at any fixed point in space,

$$\frac{\partial \phi}{\partial t} = -\,\omega \frac{\partial \phi}{\partial \theta} = \omega\, \varpi\, q_\theta \qquad [11a]$$

where ϖ denotes the distance of the point from the axis of rotation, and $q_\theta = -(1/\varpi)\, \partial\phi/\partial\theta$ and represents the transverse component of the particle velocity; see Equation [6r].

Thus Equation [9e] for the pressure p can be written, when the boundary rotates steadily,

$$p = \rho\,(\omega\, \varpi\, q_\theta - \frac{1}{2} q^2 - \Omega) + p_0 \qquad [11b]$$

or, if $\Omega = 0$,

$$p = \rho(\omega\, \varpi\, q_\theta - \frac{1}{2} q^2) + p_0 \qquad [11c]$$

where p_0 is either a constant or at most a function of the time.

These equations can also be written in terms of velocities relative to axes that rotate with the boundary. The radial component of velocity q_r' is the same as q_r, the same component taken relative to fixed axes, but the transverse component is $q_\theta' = q_\theta - \omega\varpi$. Writing $q^2 = q_r^2 + q_\theta^2$ and then substituting for q_θ,

$$p = \frac{1}{2} \rho\,(\omega^2 \varpi^2 - q'^2) + p_0 \qquad [11d]$$

where $q'^2 = q_r'^2 + q_\theta'^2$.

12. THREE-DIMENSIONAL SOURCES, SINKS, AND DIPOLES

Some important types of flow in incompressible fluid of uniform density will now be discussed briefly in preparation for the detailed studies to be made in later chapters.

Suppose that incompressible fluid is flowing radially outward from a point P with a velocity q that is a function only of the distance r from P. Then, if a sphere of radius r is drawn with its center at P, a volume $4\pi r^2 q$ of fluid flows outward from this sphere per second. Since this volume must be the same for all spheres centered at P, $r^2 q$ must be a constant, and it is possible to write

$$q = \frac{A}{r^2} \qquad [12a]$$

where A is a constant. The volume of fluid flowing outward per second from P is then $4\pi A$. A velocity potential ϕ exists; for, if

$$\phi = \frac{A}{r},$$

$$q = -\frac{\partial \phi}{\partial r}$$

At P, q is not defined and a singularity is said to occur. It may be imagined that there is a source at P in which fluid is being created at the rate $4\pi A$. In Lamb's Hydrodynamics,[1] $4\pi A$ is called the strength of the source and is denoted by m; in Milne-Thomson's Theoretical Hydrodynamics,[2] the symbol m is used for A itself and is called the strength. If A is negative, the flow is inward and a sink may be imagined to exist at P, in which fluid is being annihilated at the rate $4\pi A$. The term "source," when not specifically contrasted with "sink," will be intended in an algebraic sense, covering both sources and sinks. A flow of this type could be produced by a sphere with fixed center whose radius varies with time.

To find the distribution of pressure in the fluid, substitute in the pressure equation or Equation [9g] $q = A/r^2$ and

$$\frac{\partial \phi}{\partial t} = \frac{1}{r}\frac{dA}{dt}$$

Then

$$\frac{p}{\rho} = \frac{1}{r}\frac{dA}{dt} - \frac{A^2}{2r^4} + F(t)$$

if for simplicity p is written for p_d. At $r = \infty$, $p = \rho F(t)$. Hence if p_∞ denotes the pressure at infinity (in excess of hydrostatic pressure), assumed uniform all round,

$$p = \frac{\rho}{r}\frac{dA}{dt} - \frac{\rho A^2}{2r^4} + p_\infty \qquad [12c]$$

Other types of flow having a singularity at P can be obtained by differentiating Equation [12b], in accordance with the principle stated in Section 7. Thus, in Cartesian coordinates with origin at P, replacing ϕ by ϕ_1,

$$\phi_1 = \frac{A}{(x^2 + y^2 + z^2)^{1/2}}, \quad \frac{\partial \phi_1}{\partial x} = \frac{-Ax}{(x^2 + y^2 + z^2)^{3/2}}$$

and $\partial\phi_1/\partial x$ is also a solution of the Laplace equation. Here x can also be replaced by $r \cos \theta$ in terms of spherical coordinates with origin at P and the x-axis as axis. Thus, another solution of Laplace's equation is $(-\mu/A)(\partial\phi_1/\partial x)$ or

$$\phi = \frac{\mu x}{(x^2 + y^2 + z^2)^{3/2}} = \frac{\mu \cos \theta}{r^2} \qquad [12d]$$

where μ is a new constant.

The type of flow thus defined is said to be due to a point dipole or double source at P, also called a point doublet, because it can be produced by placing a source and sink of equal strength close together and letting their distance apart decrease to zero while the product of distance and the strength of the positive source is kept equal to μ. The line from which θ is measured is called the axis of the doublet.

13. TWO-DIMENSIONAL FLOW

The flow is two-dimensional when there is no variation of anything in a certain direction, and when the component of the velocity in that direction is everywhere zero. Thus, along any line having this direction, the pressure and the particle velocity are uniform. Each fluid particle moves in a plane perpendicular to the direction of uniformity, and the motion is the same in all of these planes. It suffices to study the motion in a single plane, which may be taken as the xy-plane. Then the z-component of the velocity w is 0, and the components u and v, like the pressure, are functions of x, y and perhaps the time t.

Alternatively, it is sometimes convenient to consider the fluid between the xy-plane and a parallel plane at unit distance from it. This part of the fluid remains permanently between the two planes, and its motion is typical of the motion of the whole.

For the two-dimensional flow of incompressible fluids it is convenient to define another function known as the stream function. Choose a fixed line perpendicular to the xy-plane, intersecting it in the point A, and a parallel line intersecting the xy-plane in P, as in Figure 10a. Let the lines be joined by an open cylindrical surface parallel to z and having the lines as two of its generators; this cylinder will intersect the xy plane in a curve, as illustrated by one of the curves in Figure 10a. Let ψ denote the volume of fluid that passes per second across the part of the cylindrical surface that lies between the xy-plane and a parallel plane unit distance away; let ψ be called positive when the fluid crosses in the positive direction of rotation about A, or in the direction from Ox toward Oy.

The quantity ψ thus defined may be described briefly as the volume of fluid that passes per second across any curve, per unit of thickness in the z-direction. Its value must be the same for all curves joining A and P, since no fluid is created or destroyed between the

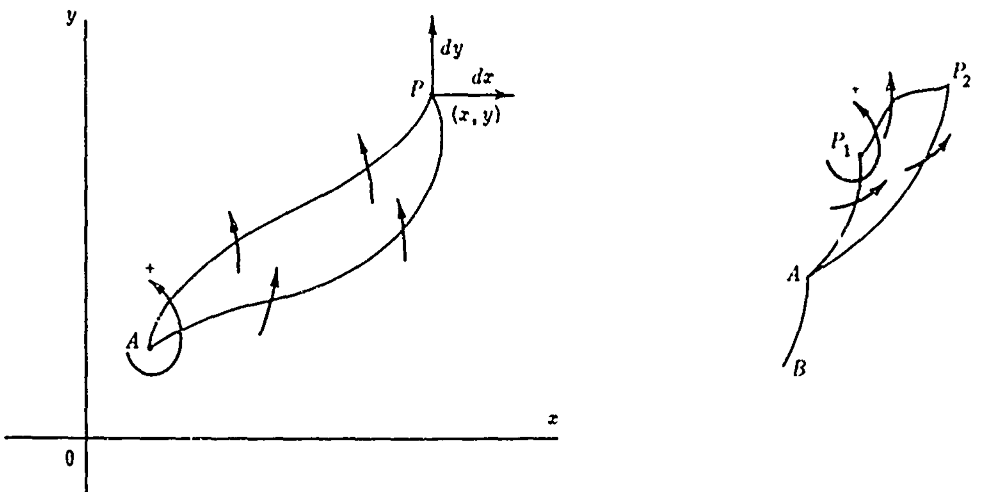


Figure 10a

Figure 10b

Figure 10 – Illustrating the definition of the stream function ψ.

corresponding cylindrical surfaces. If A is held fixed, therefore, ψ is a function of the coordinates x, y of P, and also perhaps of the time, or $\psi(x, y, t)$. This function is called the stream function. Its dimensions are those of volume per unit time per unit of length parallel to z, or L^2T^{-1}.

If the base point is moved from A to some other point B, then all values of ψ are changed by a fixed value representing the flow across BA. Thus ψ contains an arbitrary additive constant.

If the values of ψ at two points P_1, P_2 are ψ_1 and ψ_2, the rate of flow across any curve P_1P_2, as in Figure 10b, in the positive direction around P_1, per unit of length in the z-direction, is $\psi_2 - \psi_1$. If P_1 and P_2 lie on the same streamline, the rates of flow across AP_1 and across AP_2 must be the same, since there is no flow across a streamline. Hence ψ has a constant value along any given streamline. The family of curves defined by ψ = constant is thus the set of streamlines, and the streamlines themselves can be identified by means of the associated values of ψ. It follows in particular that ψ must have a constant value over any stationary boundary, which is necessarily composed of streamlines.

Simple relations exist between the stream function and the particle velocity. For, if P is displaced an infinitesimal distance dx in the x direction, ψ increases by the flow across dx or by $v\,dx$; whereas if P is displaced a distance dy in the y direction, $d\psi = -u\,dy$, in view of the convention as to the sign of ψ: see Figure 10a. Thus

$$u = -\frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x} \qquad [13a, b]$$

Or, more generally, let $\partial\psi/\partial s$ denote the space rate of variation of ψ in a chosen positive direction along any curve drawn on the xy-plane, and let q_n denote the component of the velocity normal to the curve, taken positive in a direction rotated counterclockwise through 90 degrees from the positive direction along the tangent to the curve. Thus, if $q_n > 0$, the fluid

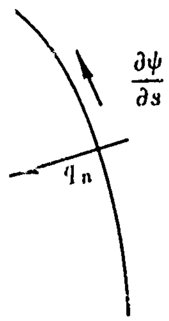


Figure 11 – Illustrating the relationship between the space rate of variation of stream function $\partial\psi/\partial s$ and particle velocity q_n.

crosses from right to left as the curve is traced positively; see Figure 11. Then

$$q_n = \frac{\partial \psi}{\partial s} \qquad [13c]$$

The stream function exists for any type of flow in an incompressible fluid, even when the motion is rotational.

If the flow is irrotational, then the velocity potential ϕ also exists, and the two families of curves, ϕ = constant and ψ = constant, cut orthogonally, since the velocity at any point is perpendicular to a curve ϕ = constant through that point and tangential to a curve ψ = constant. Then also, as in [6b, c]

$$u = -\frac{\partial \phi}{\partial x}, \quad v = -\frac{\partial \phi}{\partial y} \qquad [13d, e]$$

Comparison of these equations with Equation [13a, b] leads to the following relations between ϕ and ψ:

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \qquad [13f, g]$$

The Laplace equation for ϕ or Equation [7a] becomes, in two dimensions,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0; \qquad [13h]$$

and differentiation and subtraction of Equations [13f, g] yields also the result that

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \qquad [13i]$$

Thus in irrotational, two-dimensional flow the velocity potential and the stream function are both solutions of the two-dimensional Laplace equation. Solutions of this equation, related as stated in Equations [13f, g], are called conjugate solutions or functions. If either ϕ or ψ is known, the other can be found, except for an arbitrary constant, by integrating Equations [13e, f]

The two orthogonal families of curves, ϕ = constant and ψ = constant, are called a flow pattern. If closely spaced curves of both types are drawn, they divide the plane into small areas approximately rectangular in shape; such a diagram is called a flow net. If the same equal spacing is used for both sets of curves, the rectangles become squares; for, by the definitions of ϕ and ψ, between two adjacent ϕ curves $\delta\phi = -q\,\delta s$ where δs is the distance between them, and similarly between two ψ curves $\delta\psi = q\,\delta s'$, hence if $|\delta\phi| = |\delta\psi|$, $\delta s = \delta s'$.

This property of flow nets is sometimes made the basis of a graphical method for the construction of an approximate flow net to satisfy given boundary conditions. The ϕ and ψ curves are sketched in smoothly by estimation and are then corrected repeatedly while keeping them in harmony with the boundary conditions, until they divide the area as nearly as possible into small squares. The procedure was discussed in detail by Closterhalfen,[22] and a machine for use in such graphical constructions was described by Föttinger.[23]

Obviously ψ itself could be the velocity potential for another type of irrotational flow, satisfying a different set of boundary conditions. The stream function to accompany it would then be $-\phi$. For, if the new potential and stream function are $\phi' = \psi$, $\psi' = -\phi$, by Equations [13f, g]

$$\frac{\partial \phi'}{\partial x} = \frac{\partial \psi'}{\partial y}, \qquad \frac{\partial \phi'}{\partial y} = -\frac{\partial \psi'}{\partial x}$$

which are simply Equations [13f, g] written for ϕ' and ψ' and show that these functions stand in the relation to each other that is characteristic of a potential and its associated stream function.

Thus the two-dimensional types of irrotational flow occur in associated pairs, which might be called conjugate pairs. At a given point, velocities in two conjugate flows have perpendicular directions but equal magnitudes; in fact, the vector velocity in the second type is merely rotated, relative to that in the first type, through 90 degrees in the counterclockwise direction, or from x toward y. For, in the second type of flow the components are

$$u' = -\frac{\partial \phi'}{\partial x} = -\frac{\partial \psi}{\partial x} = -v, \qquad v' = -\frac{\partial \phi'}{\partial y} = -\frac{\partial \psi}{\partial y} = u \qquad [13j, k]$$

by Equations [13a, b]; the magnitude of the velocity is thus $q = (u^2 + v^2)^{1/2}$, and the directions are as stated, as is illustrated in Figure 12.

All of the equations written down in this section are linear and homogeneous in the dependent variables. For this reason it is easily seen that if ϕ_1, ψ_1 are the potential and stream function for one type of irrotational flow and ϕ_2, ψ_2 for another, then the sums,

$$\phi_3 = \phi_1 + \phi_2, \quad \psi_3 = \psi_1 + \psi_2$$

represent the potential and stream function for a third possible type. In the latter type, which is said to be formed by superposition of the first two, the velocity as a vector is easily seen to be the vector sum of the two component vector velocities. Again, both potential and stream function may be multiplied by the same constant.

Finally, if

$$\phi_4 = \frac{\partial \phi_1}{\partial x}, \quad \psi_4 = \frac{\partial \psi_1}{\partial x},$$

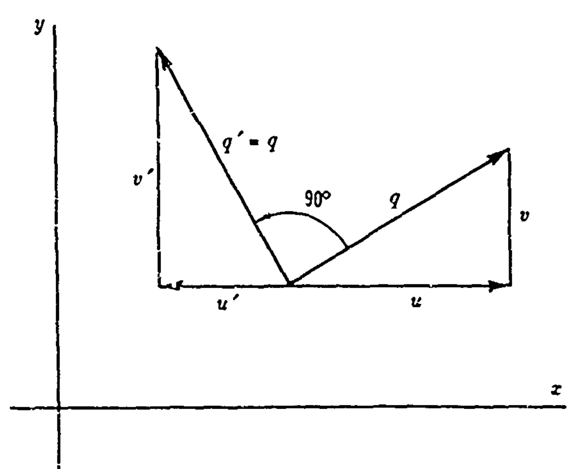


Figure 12 – Relation between particle velocities in two conjugate flows.

ϕ_4 and ψ_4 are the potential and stream function for still another possible type of flow; for, as was shown for ϕ in Section 7, ϕ_4 and ψ_4 will satisfy Equations [13h], [13i] and [13f, g]. Instead of x, y or z may be substituted in both derivatives.

It should be remarked that when the older convention mentioned in Section 6 is employed for the sign of the potential, the stream function ψ is measured by the volume of fluid crossing a curve in the clockwise direction, so that in a given case differences in the values of ψ have opposite signs, and the signs before the derivatives in Equations [13a, b] and [13j, k] are reversed. The positive direction for q_n in Equation [13c] is likewise reversed. The relation between ϕ and ψ as stated in [13f, g], however, remains the same. The simplest way to summarize the difference between the two conventions is to say that all velocities are reversed when a change is made from one to the other.

14. TWO-DIMENSIONAL FLOW IN MULTIPLY CONNECTED SPACES

In cases of two-dimensional flow, boundaries often occur which have finite dimensions in directions parallel to the planes of motion. These are called internal boundaries. They have the physical form of cylinders of unlimited length and are represented on the xy-plane by closed curves, which may or may not be circular. The presence of an internal boundary makes the space doubly connected; more generally, if two or more separate inner boundaries occur, it is multiply connected.

In irrotational motion, the circulation is required to vanish only around closed curves which do not surround any inner boundary and hence can be contracted continuously down to a point, in accordance with the explanation in Section 5. An example is curve C in Figure 13, where A represents an obstacle with a boundary that the fluid cannot penetrate. Around a curve that encircles A, such as DEF in Figure 13, the circulation may or may not vanish.

Let the positive direction along all curves that encircle boundaries be chosen in the same direction; it will be convenient to adopt the convention that, as a point traverses such a curve positively, its projection on the xy-plane eventually goes round the boundary in the counterclockwise direction, or in the direction of rotation from the x-axis toward the y-axis. Then the circulation has the same value for all closed curves that encircle A just once and do not encircle any other finite boundary.

To show this, let DEF and GHK be two such curves, and introduce a connection GD between them, as illustrated in Figure 13. Then the combined curve $DEFDGKHGD$, traced in this order, can be collapsed continuously to a point; to do this, the

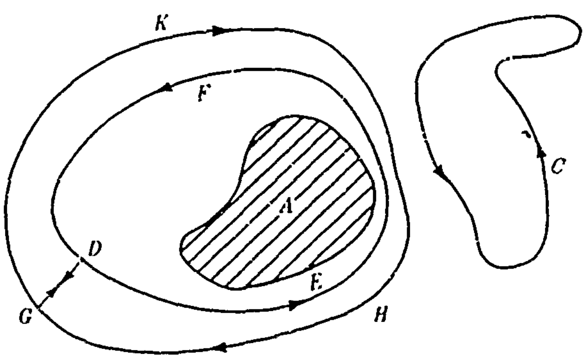


Figure 13 – Curves in a doubly connected space.

twice-traversed segment GD is first separated into two parallel segments. Then the circulation around this combined curve equals zero. But this circulation is the difference of the circulations in the positive directions about DEF and GHK; for GHK was traversed in the negative direction, so that its contribution to $\int q_s\, ds$ was reversed in sign and the contribution of GD which is traversed twice but in opposite directions, cancels out. Hence the circulations around DEF and GHK are equal.

In the same way it can be shown that the circulation about a curve that encircles several internal boundaries is the sum of the circulations around the separate boundaries.

If the velocity potential at any point P near A is now defined by

$$\phi = \int_P^{P_0} q_s\, ds$$

as in Section 6, its value for a path of integration such as PRP_0 in Figure 14 is easily seen to exceed its value for a path such as PQP_0 by the circulation Γ around A. For, geometrically, these two paths together make up a closed curve encircling A. Other paths may encircle A in the negative direction, or several times. Thus, if ϕ is the value for one path, other paths of integration may give any one of the values

$$\phi + n\Gamma$$

where n is any positive or negative integer.

The potential is thus many valued in a multiply connected space; to each point P there belongs an infinite number of values of ϕ, spaced Γ apart. The particle velocity as calculated from ϕ is, however, single-valued, since all branches of the potential, characterized by various values of n, have the same space derivatives.

If several internal boundaries are present, the potential is many valued in a more complicated fashion. In any case it follows from Equation [6g] that in going around any closed curve in the positive direction the potential decreases by an amount equal to the circulation around the curve.

An alternative prodedure sometimes adopted is to introduce enough imaginary barriers extending to infinity so that, if these barriers are never crossed by any path, the integral defining ϕ remains single-valued. Such a barrier is shown at ST in Figure 14. But then discontinuities in ϕ may occur at the imaginary barriers; and, if the velocity at a point on a barrier is to be represented by derivatives of ϕ, the barrier must be moved temporarily to one side.

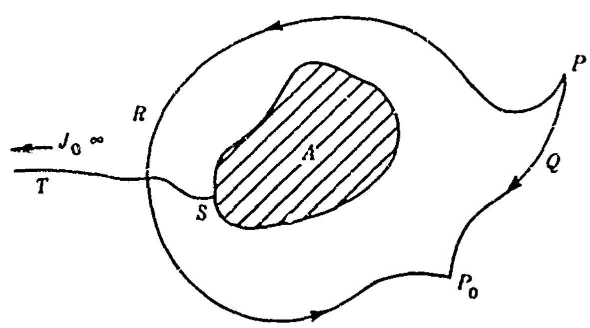


Figure 14 – Illustrating the definition of the velocity potential in a doubly connected space.

15. TWO-DIMENSIONAL OR LINE SOURCES, SINKS, VORTICES, AND DIPOLES

A two-dimensional source or uniform line source is said to exist on a line when the fluid flows uniformly away at right angles to the line. In the diagram that represents the flow in a particular plane, the line is represented by a point. By considering the flow across a circular cylinder having the line as its axis, it is easily seen that, because of the assumed incompressibility of the fluid, the velocity is

$$q = \frac{A}{\varpi} \qquad [15a]$$

where A is a constant, positive or negative, and ϖ denotes distance from the line. The volume flowing outward per unit time per unit length of the cylinder is thus $2\pi A$; either this quantity or A itself may be called the strength of the line source or sink. On the line itself the velocity is not defined.

The corresponding velocity potential is

$$\phi = -A \ln \varpi \qquad [15b]$$

where ln stands for the natural logarithm, since then $v = -\partial\phi/\partial\varpi$. It is impossible by adding a constant in ϕ to prevent it from becoming infinite at infinity; this complication in two dimensions is sometimes annoying.

The line source can also be built up by distributing infinitesimal three-dimensional point sources uniformly along the line. The constant A then represents twice the point-source strength per unit length of the line, if by source strength is meant the constant A in such formulas as [12a, b].

The potential of a two-dimensional line dipole can be obtained by differentiating that of a line source. Since $\varpi = (x^2 + y^2)^{1/2}$ in terms of Cartesian coordinates defined in a plane parallel to the flow, a possible potential for a line source at the origin is

$$\phi_1 = -A \ln (x^2 + y^2)^{1/2}$$

By differentiating ϕ_1 with respect to x and using the principles stated in Section 7, the following solution of the Laplace equation is obtained, representing a line dipole of strength μ:

$$\phi = \frac{\mu x}{x^2 + y^2} = \mu \frac{\cos\theta}{\varpi} \qquad [15c]$$

Here μ is a constant and θ is a polar angle measured from the x-axis, so that $\cos\theta = x/(x^2 + y^2)^{1/2}$.

In using this formula, irrespective of its mode of derivation, θ may conveniently be defined as e angle between two planes intersecting along a fixed line, on which the line dipole is situated, and ϖ as a coordinate representing perpendicular distance from this line; one of the planes, from which θ is measured, is fixed in position, the other rotates about the fixed line. When the motion is studied in a plane parallel to the flow, the intersection of this plane with the fixed plane is a line called the *axis* of the dipole.

As in three dimensions, the dipole can be formed by placing two simple line sources of equal and opposite strength close together and allowing them to approach each other while their strength increases without limit. It can also be formed by distributing infinitesimal three-dimensional point dipoles uniformly along the fixed line, with their axes all parallel and perpendicular to the line. The constant μ then represents twice the sum of the three-dimensional dipole moments per unit length along the line.

In Lamb's Hydrodynamics,[1] Section 60, $m/2\pi$ is written for A and $\mu/2\pi$ for μ.

A third type of flow in which a line singularity occurs is one in which, again,

$$q = \frac{A}{\varpi} \tag{15d}$$

but in which the streamlines are circles having a common axis, like the magnetic lines around a long straight current. In this case, along any one of the closed streamlines there is obviously circulation of magnitude $\Gamma = 2\pi\varpi(A/\varpi) = 2\pi A$; and it can be shown that Γ has the same value around any closed curve that encircles the axis. Around a curve that does not encircle the axis, on the other hand, $\Gamma = 0$. Thus the motion is irrotational everywhere except at points on the axis, where the velocity becomes infinite and is undefined.

Because of the resemblance of this type of flow to the motion in actual vortices, an ideal line vortex is said to exist on the axis. Its strength is measured by the circulation Γ around it. In actual "vortices" the central portion either is missing or is rotating more or less like a rigid body.

The corresponding velocity potential is discussed in Section 40.

The line dipole itself can also be interpreted as a vortex dipole, since it can be produced by allowing two vortices with equal and opposite circulations to approach coincidence while their circulations increase without limit. The axis of the resulting dipole is perpendicular to the line joining the vortices.

16. AXISYMMETRIC THREE-DIMENSIONAL FLOW

Another important case is axisymmetric flow, in which axial symmetry exists. Each particle of the fluid is confined to one of a set of fixed planes intersecting along the axis; and, at every point of any circle whose axis is the axis of symmetry, the pressure and the magnitude of the velocity have the same values and the direction of the velocity is equally inclined to the axis.

In this case, also, a stream function exists, but it is somewhat different from that for two-dimensional flow.

In any plane through the axis of symmetry, take an arbitrary but fixed point A on the axis, and any other point P joined to A by any curve AP, as in Figure 15. Consider the surface of revolution generated by the rotation of this curve about the axis. It is evident that the volume of fluid crossing this surface per second, taken positive toward the assumed negative direction along the axis, is a function only of the coordinates of P; let it be represented by $2\pi\psi$. The

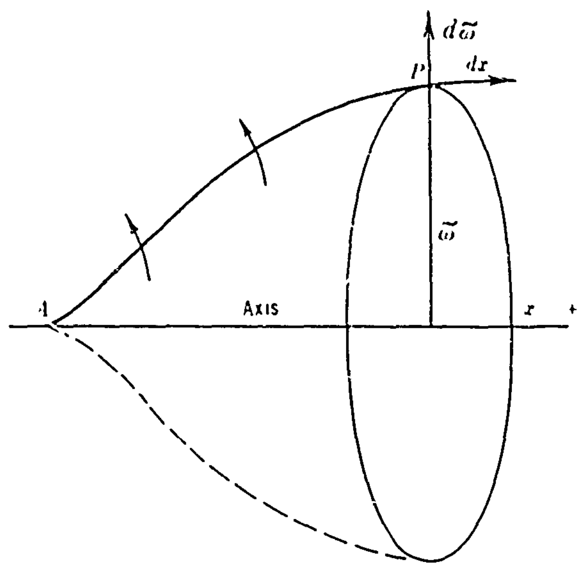


Figure 15 – Illustrating the definition of the stream function in axisymmetric three-dimensional motion.

stream function ψ thus defined, often called the Stokes stream function, represents the flow between P and the axis taken per radian of rotation about the axis.

As in two-dimensional motion, however, it is often convenient to relax the definition somewhat by adding an arbitrary constant to ψ. The dimensions of ψ are volume per unit time or L^3/t. In any plane through the axis, the curves, ψ = constant, are again the streamlines.

As coordinates, take distance x along the axis measured in the positive direction, and the distance $\tilde{\omega}$ from the axis, and let the corresponding components of the particle velocity be q_x and $q_{\tilde{\omega}}$. (Thus the x- component is denoted by u only when Cartesian coordinates are employed.) Then, if P is displaced a distance dx parallel to the axis, the flow across AP is increased by $2\pi\tilde{\omega}q_{\tilde{\omega}}dx$, and this equals $2\pi d\psi$. Or, if P is displaced a distance $d\tilde{\omega}$ outward from the axis, the flow in increased by $-2\pi\tilde{\omega}q_x d\tilde{\omega} = 2\pi\, d\psi$. Hence

$$q_x = -\frac{1}{\tilde{\omega}}\frac{\partial\psi}{\partial\tilde{\omega}}, \quad q_{\tilde{\omega}} = \frac{1}{\tilde{\omega}}\frac{\partial\psi}{\partial x} \qquad [16a, b]$$

In a similar way it can be shown that

$$q = \frac{1}{\tilde{\omega}}\frac{\partial\psi}{\partial n} \qquad [16\ ]$$

where q is the magnitude of the velocity and $\partial\psi/\partial n$ is the space rate of change of ψ in that perpendicular direction which is obtained by a *clockwise* rotation through 90 degrees from the direction of the velocity.

If a velocity potential ϕ exists, from Equations [6p, q]

$$q_x = -\frac{\partial\phi}{\partial x}, \quad q_{\tilde{\omega}} = -\frac{\partial\phi}{\partial\tilde{\omega}} \qquad [16d, e]$$

Thus ϕ and ψ are related by the equations

$$\frac{\partial\phi}{\partial x} = \frac{1}{\tilde{\omega}}\frac{\partial\psi}{\partial\tilde{\omega}}, \quad \frac{\partial\phi}{\partial\tilde{\omega}} = -\frac{1}{\tilde{\omega}}\frac{\partial\psi}{\partial x} \qquad [16f, g]$$

It is to be noted that in the axisymmetric case ϕ and ψ do not have the same dimensions.

Since x and $\tilde{\omega}$ are really Cartesian coordinates, and ϕ does not vary in the third direction, ϕ will satisfy the usual Laplace equation in terms of x and $\tilde{\omega}$ alone. The differential equation for ψ is found by substituting from [16f, g] in the identity, $\partial^2\phi/\partial x\,\partial\tilde{\omega} = \partial^2\phi/\partial\tilde{\omega}\,\partial x$.

Thus, when a velocity potential exists, ϕ and ψ are associated solutions of the two equations

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial \varpi^2} = 0, \quad \frac{\partial}{\partial x}\left(\frac{1}{\varpi}\frac{\partial \psi}{\partial x}\right) + \frac{\partial}{\partial \varpi}\left(\frac{1}{\varpi}\frac{\partial \psi}{\partial \varpi}\right) = 0 \qquad [16h, i]$$

A surface over which ψ is constant, or a stream surface, is necessarily a surface of revolution. A streamline may follow the axis up to a stagnation point, at which it divides into a sheaf of streamlines which then diverge and form a stream surface. The distance between two given stream surfaces for slightly different values of ψ varies as $1/(\varpi q)$, as is evident from Equation [16c].

The older convention as to the signs of ϕ and ψ, mentioned in Sections 6 and 13, has to be recognized again in the present connection. According to it, the signs before the derivatives would be reversed in Equations [16a, b] and [16d, e], but not in Equations [16f, g], and the direction for $\partial\psi/\partial n$ would also be reversed.

17. KINETIC ENERGY OF THE FLUID

A useful formula in terms of the potential can be obtained for the kinetic energy of the fluid. The following simple deduction may be of interest; a more rigorous proof is given in Milne-Thomson's book.[2]

Let the fluid be homogeneous and incompressible, and let it be moving with zero circulation about all closed curves. Suppose, first, that the region is enclosed within a moving finite boundary. Then the entire region can be divided up into slender tubes of flow, such that the boundary of each tube consists of streamlines. As illustrated in Figure 16, each tube must start and end on the boundary, for the reason stated in Section 8.

The kinetic energy of the fluid in unit volume is $(1/2)\rho q^2$; hence the energy in a single tube can be written

$$\delta T = \int \frac{1}{2} \rho q^2 (\delta A)\, ds \qquad [17a]$$

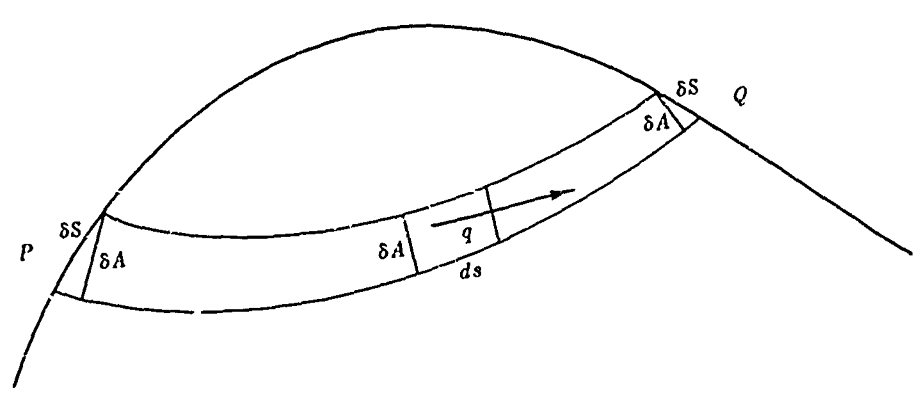


Figure 16 – Illustrating the kinetic energy of the fluid.

where q is the velocity, δA is the cross-section of the tube and ds is an element of distance along it, taken in the direction of q, so that $\delta A\, ds$ represents an element of volume. But $\rho q \delta A$ represents the mass of fluid that passes a given cross section of the tube per second and is constant along the tube, since no fluid crosses its sides. Hence $\rho q \delta A$ can be put in front of the integral sign. Furthermore, $q\, ds = -d\phi$ in terms of the velocity potential ϕ, by Equation [6f]. Hence

$$\delta T = \frac{1}{2} \rho q \delta A \int q\, ds = -\frac{1}{2} \rho q \delta A \int d\phi = \frac{1}{2} \rho q \delta A\, (\phi_P - \phi_Q) \qquad [17b]$$

where ϕ_P and ϕ_Q denote values of ϕ at the ends of the tube. Now let δS denote the element of area on the bounding surface that is enclosed by the tube, and let q_n be the component of the velocity normal to δS, taken positive toward the fluid. Then, at the end P where q_n is positive, either $q_n \delta S$ or $q\, \delta A$ represents the rate at which fluid is flowing away from the instantaneous position of δS; hence at this end $q \delta A = q_n \delta S$. At the other end, where q_n is negative, $q_n \delta S = -q \delta A$. Thus

$$\delta T = \frac{1}{2} \rho\, [(\phi\, q_n\, \delta S)_P + (\phi\, q_n\, \delta S)_Q]$$

Summation of this expression for all tubes gives for the total kinetic energy

$$T = \frac{1}{2} \rho \int \phi\, q_n\, dS = -\frac{1}{2} \rho \int \phi \frac{\partial \phi}{\partial n}\, dS, \qquad [17c]$$

in which dS stands for an element of area on the bounding surface, whereas q_n represents the component of velocity normal to the boundary, taken positive toward the fluid, and equals $-\partial\phi/\partial n$ by Equation [6f], where ∂s is replaced by ∂n, representing an elementary displacement away from the boundary and along the normal. The integral extends over the entire boundary.

For *two-dimensional* motion, let T_1 denote the kinetic energy of the fluid between two planes drawn parallel to the planes in which the particles move, and unit distance apart. The integral in Equation [17c] may then be taken only over the included part of the boundary; and, since the motion is the same in all planes, dS may be given the form of a strip of unit length and width ds, where ds is an element of distance along the curve representing the boundary in a typical plane. Thus, provided there is no circulation,

$$T_1 = \frac{1}{2} \rho \oint \phi\, q_n ds = -\frac{1}{2} \rho \oint \phi \frac{\partial \phi}{\partial n}\, ds = \frac{1}{2} \rho \oint \phi\, d\psi \qquad [17d]$$

Here the curve is assumed to be traversed with the fluid lying on the left; $\partial\phi/\partial n$ is the space rate of change of ϕ toward the fluid along the normal to the boundary, and the last expression results from Equation [13c] and the relation $(\partial\psi/\partial s) ds = d\psi$.

If circulation is present the formula for T_1 must be modified. In the case of a stationary cylinder inside a stationary cylindrical shell, only circulatory flow is possible, and the tubes of flow are all closed on themselves. Here, in Equation [17b], P and Q coincide and $\phi_P - \phi_Q = \Gamma$, the circulation, which is the same for all tubes: also, $\Sigma q \delta A$, summed for all tubes between two planes unit distance apart, equals $\psi_s - \psi_c$ where ψ_s is the value of the stream function on the shell and ψ_c its value on the enclosed cylinder. Hence in this case

$$T_1 = \frac{1}{2}\rho\Gamma(\psi_s - \psi_c) \qquad [17e]$$

If the cylinder and shell are in motion, it is only necessary to superpose upon the circulatory flow F_2 as just described another flow F_1 having a single-valued potential ϕ_1 such as is caused by the motion of the boundaries when $\Gamma = 0$. The kinetic energies associated with these two flows are simply additive, giving a total of

$$T = \frac{1}{2}\rho\int \phi_1\, q_n\, ds + \frac{1}{2}\rho\,\Gamma\,(\psi_{2s} - \psi_{2c}) \qquad [17f]$$

where the first integral extends around both shell and cylinder and ψ_2 is the stream function for F_2 alone. For, if at any point in the fluid the particle velocity due to ϕ_1 has a component q_{1n} normal to the direction of the velocity q_2 due to F_2, and a component q_{1p} parallel to q_2, then

$$q^2 = q_{1n}^{\;2} + (q_{1p} + q_2)^2 = q_1^{\;2} + q_2^{\;2} + 2q_{1p}\,q_2$$

If, now, $\rho\, q_{1p}\, q_2\, \delta A$ is integrated along any tube of F_2, just as $\rho\, q^2/2$ was in Equation [17b], $\rho q_2 \delta A$ is again constant, and $\int q_{1p} ds_q = -$ (change in ϕ_1) $= 0$. Thus the product term $q_{1p}q_2$ contributes nothing on the whole to the kinetic energy.

If several cylinders are present inside the shell, the flow can be resolved into F_1 and a number of circulatory flows, in each of which there is the same circulation about all paths encircling a certain one of the cylinders once and zero circulation about all paths not encircling it. Then the argument can be extended to prove that the total kinetic energy is simply the sum of the energies associated with each of these component flows.

In the *axisymmetric* case, the element of area dS may take the form of a ring cut out of the bounding surface by two neighboring planes perpendicular to the axis; see Figure 17. The width of the ring is the length ds of the arc that is cut out by the planes from the curve representing the boundary on a typical plane through the axis, and its perimeter is $2\pi\varpi$ where ϖ denotes distance from the axis; hence its area is $dS = 2\pi\varpi ds$. Thus, from Equation [17c],

$$T = \pi\rho \int \phi\, q_n\, dS = \pi\rho\int \phi\, d\psi \qquad [17g]$$

after substituting $dS = 2\pi\varpi ds$, $q_n = (1/\varpi)\,\partial\psi/\partial n$ from Equation [16c], $\partial\psi/\partial n = \partial\psi/\partial s$, and $(\partial\psi/\partial s)\, ds = d\psi$.

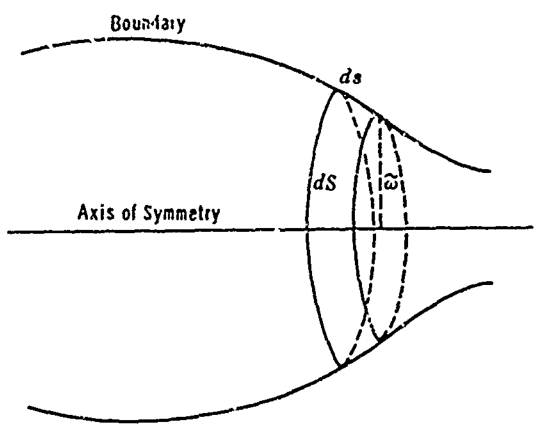


Figure 17 – Illustrating the kinetic energy of the fluid for an axisymmetric surface.

These formulas can all be shown to hold also for an infinite mass of fluid surrounding a moving internal boundary, provided the velocity vanishes at infinity and provided there is no circulation. Here the integral is taken only over the internal boundary.

Two-dimensional circulatory flow about an internal boundary immersed in infinite fluid leads to an infinite value for T_1. This circumstance, although inconvenient, does not invalidate other conclusions from the theory, since a cylinder of infinite length is in any case an abstraction introduced in order to simplify the mathematics.

18. UNITS OF MEASUREMENT

In all cases, a consistent set of dynamical units is assumed to be employed. In using each formula, any unit of length may be used, but the same unit must be used for all linear dimensions, the square of that unit must be used for areas, and the cube for volumes. A common unit of time must be employed for all velocities and accelerations.

If forces are measured in pounds, time in seconds, and linear dimensions in feet, then pressure is in pounds per square foot; mass is measured in slugs, or pounds times seconds squared divided by feet, and equals weight in pounds divided by the acceleration of gravity or by 32.2; density is in slugs per cubic foot; energy is in foot-pounds.

If forces are measured in pounds and time in seconds, but linear dimensions in inches, then pressure is in pounds per square inch; mass is in pounds times seconds squared divided by inches and equals weight in pounds divided by 386, which is the acceleration due to gravity expressed in inches per second squared; density equals pounds per cubic inch divided by 386; energy is in inch-pounds.

The velocity potential has the dimensions of velocity multiplied by distance; hence it will be in feet squared divided by seconds if lengths are expressed in feet and time in seconds, or in inches squared divided by seconds if inches are substituted for feet. The same units apply to the circulation as to the velocity potential.

Angles may always be measured in radians, and this unit is always understood when an angle is added or equated to a quantity that is not an angle, as in Equation [138f′] in Section 138; this holds whether the angle is represented by a single symbol, such as θ, or indirectly by a symbol such as $\sin^{-1}$. In equations between angles, like Equation [38b], or when a trigonometric function such as $\sin\theta$ is indicated, degrees may be used instead of radians.

It may be remarked that the symbol $\sqrt{P}$ will be used to denote the positive square root of any expression P whenever P represents a positive real number; and such angles as $\sin^{-1}x$ or $\tan^{-1}x$ will be understood to be in the first quadrant whenever x is so limited by the circumstances of the case that this interpretation cannot fail. Otherwise these symbols are to be interpreted as many-valued except insofar as a special rule is stated for their interpretation.

CHAPTER II

THE USE OF COMPLEX FUNCTIONS IN HYDRODYNAMICS

In two-dimensional hydrodynamics extensive use is made of functions of a complex variable. In this chapter, therfore, the mode of application of the theory of complex variables in hydrodynamics will be discussed, and a summary will be included of the principal relevant parts of the mathematical theory.

For convenience of reference, a short table of formulas pertaining to the hyperbolic functions is appended; and some useful series are also listed.

19. COMPLEX NUMBERS

The so-called imaginary numbers were invented in order to solve certain algebraic equations, such as $x^2 = -1$. A solution of this equation is $x = i$, where i is a symbol having the property that $i^2 = -1$. In other respects i is assumed to behave like a real number. Obviously $i^3 = ii^2 = -i$, $i^4 = (i^2)^2 = 1$, $1/i = -i$. The product of i by a real number is called an imaginary number.

The sum of a real number and an imaginary number is called a complex number; it can be written

$$z = x + iy$$

where x and y are real numbers.

The number $z^* = x - iy$ is formed from z by changing the sign of the imaginary part and is called the complex conjugate of z. It is often denoted by $\bar{z}$.

Complex numbers are conveniently represented on a plot called the Argand diagram. In this plot the real part x is plotted as abcissa and the imaginary part y with i omitted is plotted as ordinate, as in Figure 18. Either the point (x, y) or the vector drawn from the origin to this point may be regarded as representing the complex number.

In labeling points and lines on such diagrams, it is convenient sometimes to use special symbols representing geometrical quantities only, and sometimes to use symbols that stand for numbers, real or complex. This leads to no difficulty in spite of the logical difference between geometrical magnitudes on a plane and complex numbers.

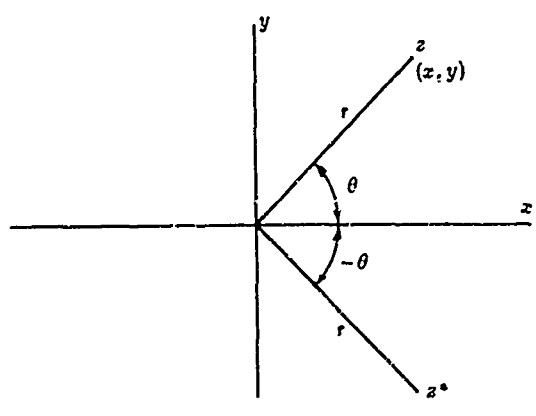


Figure 18 – Argand diagram.

It is often convenient to express a complex number in terms of the polar coordinates

$$r = \sqrt{x^2 + y^2} \geq 0, \quad \theta = \tan^{-1} \frac{y}{x}$$

The geometrical significance of r and θ on the plot is shown in Figure 18. Here r is called the *modulus* or absolute value of z and is denoted by $|z|$ or mod z. The angle θ is called the *amplitude*, or sometimes the argument of z; it is denoted by amp z, or arg z.

The amplitude is multiple-valued, since if θ_1 is one value, another possible value is $\theta_1 + 2n\pi$ where n is any positive or negative integer. A complex number is completely specified when its modulus and amplitude are given; for, in terms of r and θ,

$$z = x + iy = r(\cos\theta + i\sin\theta)$$

In referring to the amplitude, however, it is often necessary to specify which of its many values is meant. The value of the amplitude θ such that

$$-\pi < \theta \leq \pi$$

is called its principal value; this value is often tacitly understood. It should be carefully noted that no ambiguity attaches to the value of the complex number itself; the ambiguity attaches only to its polar representation.

Numbers for which $r = 1$ are represented on the diagram by points lying on the unit circle, or a circle about the origin of radius unity.

Two complex numbers are equal only when their real parts are equal and their imaginary parts are also equal. For this reason every equation between complex numbers is equivalent to two real equations, one containing the real parts, and the other the imaginary parts with i omitted. Equal complex numbers have equal moduli, and their amplitudes can differ only by an integer multiplied by 2π.

In the diagram, the sum of two complex numbers is represented by the vector sum of the vectors representing the two numbers, as in Figure 19. For, if

$$z_1 = x_1 + iy_1, \quad z_2 = x_2 + iy_2,$$

then

$$z_1 + z_2 = x_1 + x_2 + i(y_1 + y_2)$$

It should be noted that the amplitude of the sum or difference of two numbers is not uniquely fixed by an assignment of the amplitudes of the two numbers; it is partly arbitrary and must be separately chosen if needed.

The product, on the other hand, has nothing to do with the ordinary vector products of the corresponding vectors. Multiplication and division can be done in cartesian form, thus:

$$z_1 z_2 = x_1 x_2 - y_1 y_2 + i(x_1 y_2 + x_2 y_1),$$

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i\,\frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}$$

In the latter formula the separation of the quotient into real and imaginary parts has been effected by the usual and important device of rationalizing the denominator, that is, both numerator and denominator are multiplied by the complex conjugate of the denominator or $x_2 - iy_2$.

Substitution of

$$x_1 = r_1 \cos \theta_1, \quad y_1 = r_1 \sin \theta_1,$$

$$x_2 = r_2 \cos \theta_2, \quad y_2 = r_2 \sin \theta_2$$

gives, after some trigonometric substitutions,

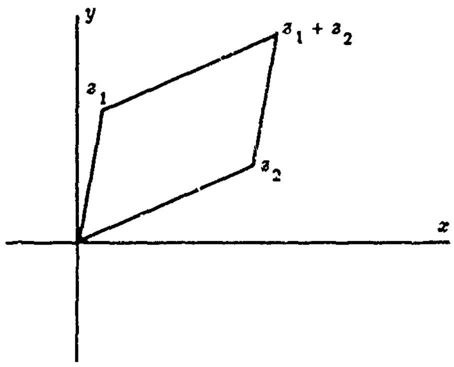


Figure 19 – The addition of two complex numbers z_1 and z_2.

$$z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)].$$

Thus the modulus of the product is the product of the moduli of the factors; and the amplitude of the product is naturally obtained as the sum of the amplitudes. Similarly, the modulus of the quotient is the quotient of the moduli; whereas the amplitude of the quotient may be taken to be the difference between the amplitudes of numerator and denominator. This convention as to amplitudes of product and quotient will be retained throughout. An example is illustrated in Figure 20.

Multiplication of a complex number by i merely increases its amplitude by $\pi/2$, or rotates the representative vector on the diagram counterclockwise through $\pi/2$. Multiplication by $-i$ decreases the amplitude by $\pi/2$ and rotates the vector clockwise.

The following formulas may be noted:

$$zz^* = (x + iy)(x - iy) = x^2 + y^2 = r^2;$$

$$z + z^* = 2x, \quad z - z^* = 2iy;$$

$$|z^n| = |z|^n \text{ for real } n, \quad |z_1 z_2| = |z_1|\,|z_2|;$$

$$|z_1/z_2| = |z_1|/|z_2|$$

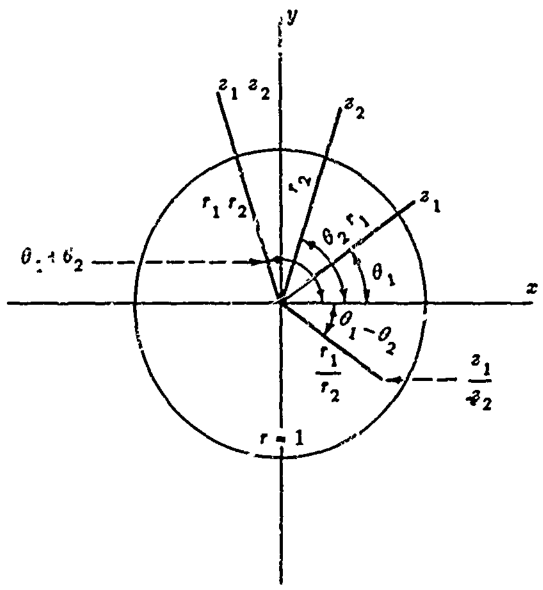


Figure 20 – Illustrating the product and quotient of z_1 and z_2.

20. SOME COMMON FUNCTIONS OF z

Functions constructed by means of algebraic processes, perhaps with the addition of the process of taking a limit, can be taken over at once from the field of real numbers into the complex field. It is convenient to begin with certain transcendental functions defined by means of series.

In the defining series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \qquad [20a]$$

substitution of $i\theta$ for x gives

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} \qquad [20b]$$

Comparison with the series for sin θ and cos θ, which are stated in Equations [33b, c] shows that the following important formula holds:

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad [20c]$$

Thus

$$e^{i\pi/2} = i, \quad e^{-i\pi/2} = -i, \quad e^{\pm i\pi} = -1$$

It follows that any complex number can be written in the alternative forms

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

Its conjugate is

$$z^* = x - iy = r(\cos\theta - i\sin\theta) = re^{-i\theta}$$

Two other useful functions are the hyperbolic sine and cosine:

$$\sinh x = \frac{1}{2}(e^x - e^{-x}) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots \qquad [20d]$$

$$\cosh x = \frac{1}{2}(e^x + e^{-x}) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots \qquad [20e]$$

From the series it is easily verified that

$$\sin(iy) = i\sinh y, \quad \cos(iy) = \cosh y,$$

$$\sinh(iy) = i\sin y, \quad \cosh(iy) = \cos y$$

Finally, writing $z = re^{i\theta}$ and ln for the natural logarithm,

$$\ln z = \ln r + i\theta = \tfrac{1}{2}\ln(x^2 + y^2) + i\tan^{-1}\frac{y}{x}$$

Here ln r or $\ln(x^2 + y^2)$ is to be interpreted as the ordinary real logarithm. Thus ln z is many-valued. Its imaginary part has an infinite number of values spaced $2\pi i$ apart, namely, written in terms of any one of them $i\theta$, $i\theta + 2\pi i$, $i\theta + 4\pi i \ldots$, $i\theta - 2\pi i$, $i\theta - 4\pi i \ldots$ Even if $z = x$ and is real and positive, for complete generality $\ln z = \ln x = (\ln x)$ real $+ 2\pi ni$, where n is any integer, positive or negative.

21. POWERS OF z

Writing $z = re^{i\theta}$

$$z^n = r^n e^{in\theta} = r^n(\cos n\theta + i \sin n\theta)$$

by [20c]. If n is an integer, positive or negative, z^n is single-valued; in particular, $z^0 = 1$, as for a real number. For nonintegral n, z^n is many-valued, because of the ambiguity of θ.

For example:

$$z^{1/2} = r^{1/2}\left[\cos\left(\frac{1}{2}\theta\right) + i \sin\left(\frac{1}{2}\theta\right)\right]$$

But, if θ is replaced by $\theta + 2m\pi$ where m is an integer

$$z^{1/2} = r^{1/2}\left[\cos\left(\frac{1}{2}\theta + m\pi\right) + i \sin\left(\frac{1}{2}\theta + m\pi\right)\right]$$

If m is even, the expression in brackets reduces to $\cos(1/2\ \theta) + i \sin(1/2\ \theta)$ and the same value of $z^{1/2}$ is obtained as before. If, however, m is an odd integer, positive or negative,

$$z^{1/2} = -r^{1/2}\left[\cos\left(\frac{1}{2}\theta\right) + i \sin\left(\frac{1}{2}\theta\right)\right]$$

Thus, as for a real number, $z^{1/2}$ has two values, each the negative of the other; see Figure 21.

Similarly, $z^{1/3}$ has three values, with applitudes spaced $2\pi/3$ radians apart; and, in general, if k is a positive integer, $z^{1/k}$ has k different values with amplitudes spaced $2\pi/k$ apart. If n is not a rational number, that is, the ratio of two integers, z^n has an infinite number of values.

In working with many-valued functions such as $\ln z$ or z^n, the value that is to be employed for *amp* z must be clearly established. If possible, *amp* z is usually so chosen that the given function varies continuously as z is allowed to vary through such sets of values as are of interest, and is also continuous with the same function as ordinarily understood when z becomes real and positive. Thus, for real $z > 0$, $\ln z$ is made to become the ordinary real $\ln z$, and z^n is real and positive.

Special care is needed when a more complicated function of z is involved, as, for example, in $\ln[(z+a)/(z+b)]$. Every sum or difference represents a new entity for which a special rule must be adopted for the determination of its amplitude. Algebraic changes are

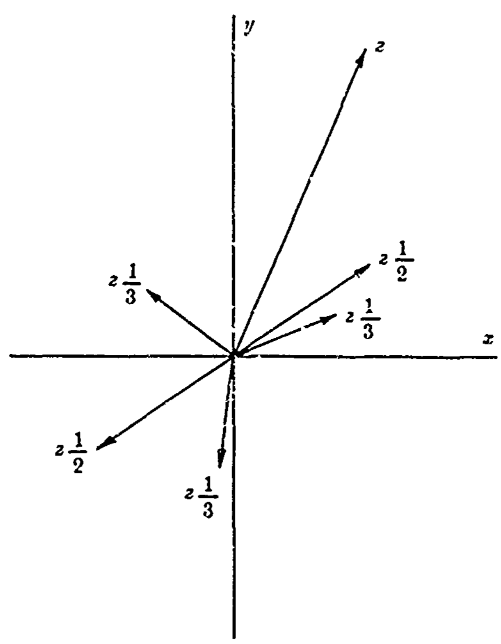


Figure 21 - The various values of $z^{1/2}$ and $z^{1/3}$.

treacherous; for example, $\ln [z^2/(-z)] \neq \ln (-z)$, even $\ln [1/(-1)] \neq \ln (-1)$, if the same rule is used for amp $(-z)$ or amp (-1) in both places. The safest procedure here is to calculate $\ln z$ from the amplitudes and absolute values of the separate factors.

22. REGULAR FUNCTIONS OF A COMPLEX VARIABLE

If, in the complex number, $z = x + iy$, x and y are allowed to vary, z becomes a complex variable. If a value of another complex variable w is associated by means of some rule with each value of z, then w is a function of z. It is also possible to regard w as a complex function of the real variables x and y, of which its real and imaginary parts are likewise functions. Thus

$$w = f(z) = g(x,y) = \phi(x,y) + i\psi(x,y) \qquad [22a]$$

where ϕ and ψ are real functions of x and y.

Some functions of z are single-valued, that is, there is only one value of the function associated with each value of z; others are many-valued. A function which, at all points within a certain region on the z-plane, is both single-valued and differentiable, is said to be *regular* or *analytic* or holomorphic within that region.* Such a function is also said to be regular or analytic or holomorphic at any point in the interior of the region. Many functions are regular except at certain points called singular points.

In dealing with many-valued functions, a particular branch of the function can often be defined so that, taken by itself, it is regular within a certain region. Thus, if $z = re^{i\theta}$ and θ is kept within the range $-\pi < \theta \leq \pi$, $\ln z$ is an analytic function of z, except where $\theta = \pi$, since at such points $\ln z$ cannot be differentiated without overstepping the bounds set for θ.

It can be shown that regular frunctions necessarily posses derivatives of all orders.The reason for this special behavior lies in the fact that a point on the z-plane can be approached from many different directions. Thus, in the formula

$$\frac{df}{dz} = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z}$$

if z is a real variable, Δz can vary only in magnitude, whereas if z is a complex variable the increment Δz may vary also in amplitude, or in the direction of the representative vector on the diagram; nevertheless the limit is required to have a fixed value, real or complex. This requirement imposes a severe restriction upon the behavior of the function.

The existence of a derivative with respect to z requires, in fact, that certain differential equations in terms of x and y must be satisfied. Consider

$$f(z) = f(x+iy) = \phi(x,y) + i\,\psi(x,y)$$

where ϕ and ψ are real. Regarding $f(z)$ on the one hand as a function of z and on the other hand as a function of x and y, by the ordinary rule for the differentiation of a function of a function,

*Some writers call a function analytic within a region when it has the properties stated except at a finite number of singular points.

$$\frac{\partial f}{\partial x} = \frac{df}{dz}\frac{\partial z}{\partial x} = \frac{df}{dz}, \qquad \frac{\partial f}{\partial y} = \frac{df}{dz}\frac{\partial z}{\partial y} = i\frac{df}{dz}$$

Hence

$$\frac{df}{dz} = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$$

or

$$\frac{df}{dz} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = -i\frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial y} \qquad [22b]$$

and, equating real and imaginary parts separately in this last equation,

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \qquad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \qquad [22c,d]$$

These equations are known as the Cauchy-Riemann relations. They hold necessarily wherever $f(z)$ is differentiable; and it can be shown that they guarantee the existence of a derivative with respect to z wherever the derivatives of ϕ and ψ are continuous functions of x and y.

From [22b] and [22c,d] it follows that

$$\left|\frac{df}{dz}\right|^2 = \left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 = \left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2 \qquad [22e]$$

where $|\ |$ denotes as usual the absolute value.

If $w = f(z)$, then $z = F(w)$ where F denotes another function known as the inverse of the function f. If $f(z)$ is a regular function at any point z, so is $F(w)$ at the corresponding value of w. As in real variables,

$$\frac{dz}{dw} = \left(\frac{dw}{dz}\right)^{-1}$$

23. CONFORMAL REPRESENTATION OR MAPPING

Assume that $z = x + iy$

and

$$w = f(z) = \phi + i\,\psi \qquad [23a]$$

where $f(z)$ denotes a regular function of z and ϕ and ψ are real functions $\phi(x,y)$, $\psi(x,y)$. Suppose that the values of w are plotted on the same plane with those of z, with a common real axis. Then the transformation from z to w displaces each point on the plane, representing a value of z, into another position where it represents a value of w. Curves are displaced and, in general, changed in shape.

Often, however, it is more convenient to plot w on a separate plane called the w-plane. Then, to each point or curve on one plane there corresponds a point or curve on the other. The configuration on the z-plane is said to be transformed into that on the w-plane, or to be represented by it, by means of the transformation $w = f(z)$. A diagram on the w-plane can be regarded as a kind of map of the corresponding z diagram. The comparison is facilitated if the two planes are thought of as parallel, and with parallel axes for real and imaginary numbers.

Corresponding curves on the two planes will usually differ both in linear scale and in direction. Let z undergo a small increment δz along a curve, as from P_1 to P_2 in Figure 22.

Then w will receive a small increment $Q_1 Q_2$ or

$$\delta w = \frac{dw}{dz} \delta z$$

Here dw/dz and δz are complex numbers; they can be written

$$\frac{dw}{dz} = Re^{i\alpha}, \; \delta z = |\delta z| \, e^{i\beta}$$

in terms of real numbers R, α, $|\delta z|$ and β. Then

$$\delta w = R \, |\delta z| \, e^{i(\beta + \alpha)}$$

Suppose that $R \neq 0$ so that $dw/dz \neq 0$. Then this last equation shows that the line element δw can be formed out of the line element δz by first stretching or shrinking it in the ratio represented by R, or by the modulus of dw/dz, and then rotating it through the angle α, which is the amplitude of dw/dz.

Thus the vector representing δw makes with the axis of reals on the w-plane an angle greater by α than the angle that the vector representing δz makes with the real axis on the z-plane; see Figure 22. Any other line element at P_1, such as $P_1 P_3$, is changed in scale in the same ratio and is rotated through the same angle and in the same direction. The derivative dw/dz can be thought of as an operator that transforms the line elements in this manner; it stretches the local area in the ratio R and rotates it through the angle α.

It follows that, if the two curves interesect at an angle γ on the z-plane, the transformed curves will intersect at the same angle γ on the w-plane. Furthermore, the angle is not turned over; a rotation in the same direction through an angle γ swings the tangent from one curve to the other on either plane. Thus a transformation by means of a regular function $f(z)$ completely preserves the angles between intersecting curves at all points at which $df/dz \neq 0$. Infinitesimal figures also keep the same shape, although they may be changed in scale and rotated through a certain angle, without being turned over.

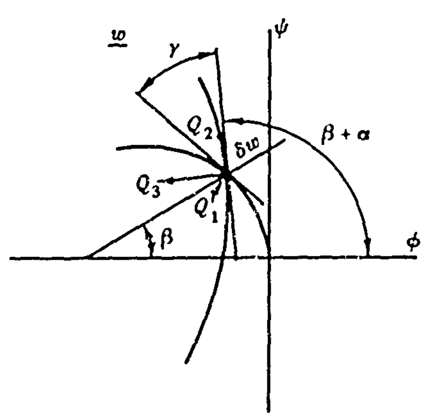


Figure 22a

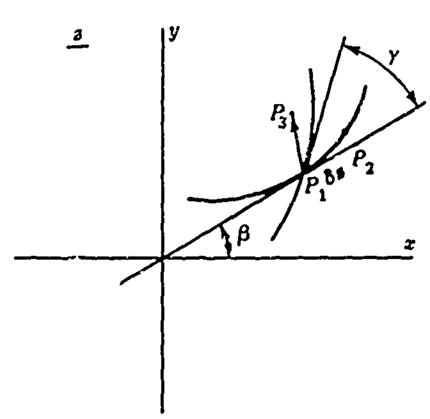


Figure 22b

Figure 22 – Illustration of conformal mapping.

A transformation or representation which preserves angles and the shape of infinitesimal figures is called isogonal; if the angles are also not turned over, it is called conformal. Mercator's projection represents a conformal mapping of the earth's surface on a plane.

A figure of finite size, however, does not usually retain its shape under a transformation, since the change of scale and the rotation are usually different at different points, because of variation in the value of dw/dz.

The angle between two curves may fail to be preserved if they intersect either at a singular point, where df/dz does not exist, or at a point at which $df/dz = 0$.

It should be noted that as δz is rotated in direction, by changing its amplitude, δw rotates in the same direction, and by an equal amount. Hence, as the z-point traverses a curve in a certain direction and the w-point traverses the corresponding curve on the w-plane, the area on the left-hand side of one curve corresponds to that on the left-hand side of the other, and the area on the right of one to the area on the right of the other. For example, in Figure 23, rotating δz as shown off the curve and toward the area S causes δw to rotate toward the area T; this shows that points lying near the curve and in S transform into points in T. Similarly, nearby points in U transform into points in V. This rule is very useful in the study of conformal mapping.

The transformation can also be viewed from the inverse standpoint, as a mapping of the w-plane on the z-plane by means of the inverse transformation,

$$z = F(w)$$

where F is the inverse function obtained by solving Equation [23a] for z. Then

$$x = x(\phi,\psi), \quad y = y(\phi,\psi)$$

Two families of curves on the z-plane that are of particular interest are those defined by $\phi(x,y) =$ constant and $\psi(x,y) =$ constant. From the conformal property of the transformation, it follows that these two families of curves intersect orthogonally, wherever dw/dz is finite and not zero, as illustrated in Figure 29, page 48. For, this is obviously true of the corresponding curves on the w-plane, which are straight lines parallel to the axes. The orthogonality can also be verified directly from Equations [22b,c].

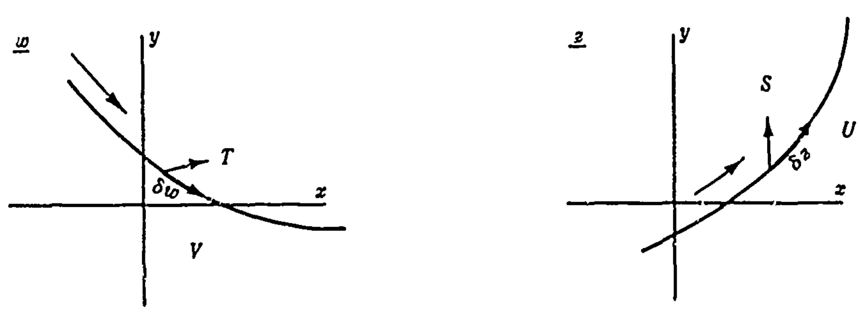


Figure 23 – The correspondence of regions adjoining a curve in conformal mapping.

The values of ϕ and of ψ that are thus associated with each point on the z-plane can be employed as curvilinear orthogonal coordinates on that plane. Thus each regular transformation furnishes in ϕ and ψ a special set of orthogonal coordinates.

If $w = f(z)$ is a many-valued function but such that any one branch of it, taken by itself, has a unique derivative, then each branch maps an area of the z-plane onto the w-plane, independently of all other branches.

Finally, a device pointed out by Maxwell may be mentioned that is sometimes useful in drawing the curves. Suppose the curves, ψ = constant, are to be drawn, and that ψ is the sum of two terms:

$$\psi(x,y) = f(x,y) + g(x,y)$$

First draw the two sets of curves, $f(x,y)$ = constant, $g(x,y)$ = constant, using the same equal spacing for the constant values of f and g. These curves divide the plane into approximate parallelograms. Then it is easily seen that curves, $\psi(x,y)$ = constant, for equally spaced values of ψ, pass through opposite corners of these parallelograms as illustrated in Figure 24,

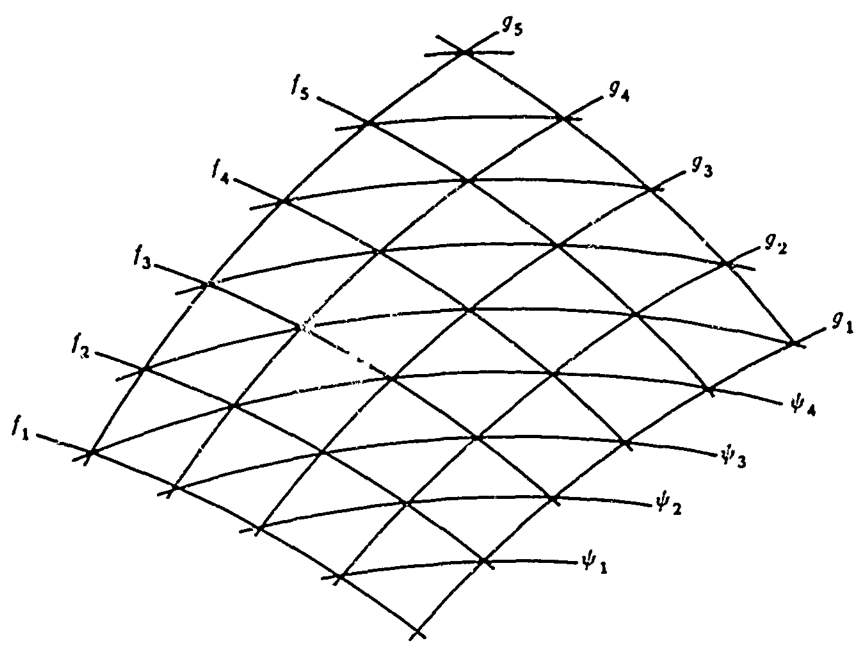


Figure 24 – Maxwell's construction for curves defined by the sum of two functions.

24. EXAMPLES OF CONFORMAL TRANSFORMATIONS

(1) Consider first the linear transformation

$$w = Az + B$$

where A and B are fixed numbers, perhaps complex. Let $A = a + ib$ where a and b are real. Then

$$\frac{dw}{dz} = A = \sqrt{a^2 + b^2}\, e^{i \tan^{-1}(b/a)}$$

Since dw/dz does not vary from point to point of the z-plane, even finite figures will transform conformally under this transformation; their linear dimensions, however, will be changed on the w-plane in the ratio $\sqrt{a^2 + b^2}$ and they will be rotated, relatively to the real axis, through an angle equal to amp A or to $\tan^{-1}(b/a)$. They will also be displaced in the direction of the vector representing B.

(2). Another interesting transformation is the inverse transformation

$$w = \frac{1}{z}$$

or

$$w = \frac{1}{r} e^{-i\theta}, \quad z = re^{i\theta}$$

The transformation from the z-plane to the w-plane may be imagined to be made in two steps. First, let each point P at r distance from the origin of z be moved to a position P' lying on the same radius from the origin but at a distance $1/r$; that is, each point is displaced to its inverse point in the unit circle, $r = 1$. Such a geometrical transformation is called inversion with respect to the circle. It can be visualized by imagining the plane to be turned inside out while the unit circle stands still. Then let each point be moved to its mirror image in the real axis; this changes the sign of θ. Thus the inverse transformation is equivalent geometrically to inversion in the unit circle plus a reflection in the real axis. These two steps may be taken in either order. The changes may be imagined to be executed on the z-plane, which is then rechristened the w-plane; see Figures 25 and 26.

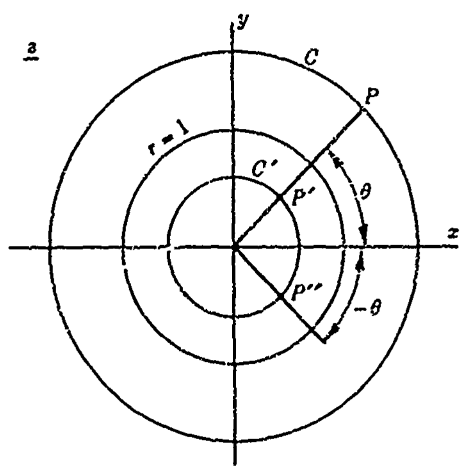


Figure 25 – The transformation $w = 1/z$

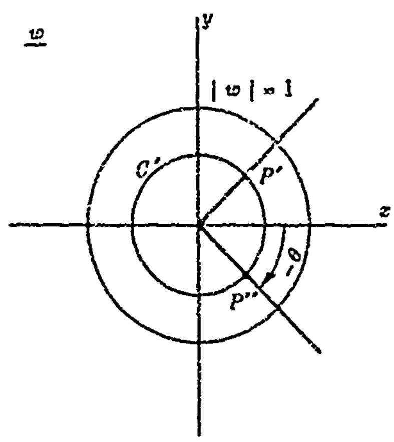


Figure 26 – The transformation $w = 1/z$.

The exterior of the unit circle on the z-plane is thus mapped onto the interior of this circle on the w-plane, and vice versa. Radial lines transform into radial lines with reflection in the real axis.

It is easily shown that a circle passing through the origin transforms into a straight line not passing through the origin, whereas any other circle transforms into a circle. If the circle is centered at the origin, so is the transformed circle, but they lie in inverse positions with respect to the unit circle.

It is sufficient to prove these statements for the inversion. Referring to Figure 27, for P on a given circle C,

$$r^2 + h^2 - 2\,hr\cos\theta = a^2.$$

The result of substituting $r = 1/r_1$ where r_1 is the value of r at the inverse point P_1, may be written

$$r_1^2 + \frac{h^2}{(h^2 - a^2)^2} - 2\,\frac{hr_1}{h^2 - a^2}\cos\theta = \frac{a^2}{(h^2 - a^2)^2}$$

which locates P_1 similarly on another fixed circle. If $h = a = b$ as for C' in the figure, $r = 2b\cos\epsilon$, $r_1\cos\epsilon = 1/(2b)$ where $r_1 = 1/r$, so that P_1 is located on the line ST.

The point $z = 0$ is a singularity to which, in strictness, the transformation does not apply. It is often convenient, however, to speak of a single "point at infinity." If this is done, it can be said that the transformation $w = 1/z$ transforms the point $z = 0$ into the point $w = \infty$. If z is allowed to approach $z = 0$ in a certain direction, w recedes toward ∞ in a corresponding direction, and vice versa. If z goes around $z = 0$ along a curve of very small diameter, w goes around $w = \infty$ along a curve on which $|\,w\,|$ is everywhere large, and vice versa.

The transformation $w = 1/z$ is single-valued in both directions; any point of the z-plane is transformed into a definite point on the w-plane, and the inverse transformation $z = 1/w$ transforms any point on the w-plane into a definite point on the z-plane.

Further formulas for this transformation will be found in Section 37.

The most general transformation that transforms all lines and circles into lines or circles is the bilinear transformation, sometimes called linear, or

$$w = \frac{az + b}{cz + d}$$

where a,b,c, and d are constants, real or complex

(3). The Transformation

$$w = z^{1/2}$$

on the other hand, is double-valued, transforming every point on the z-plane except 0 and ∞ into two points on the w-plane.

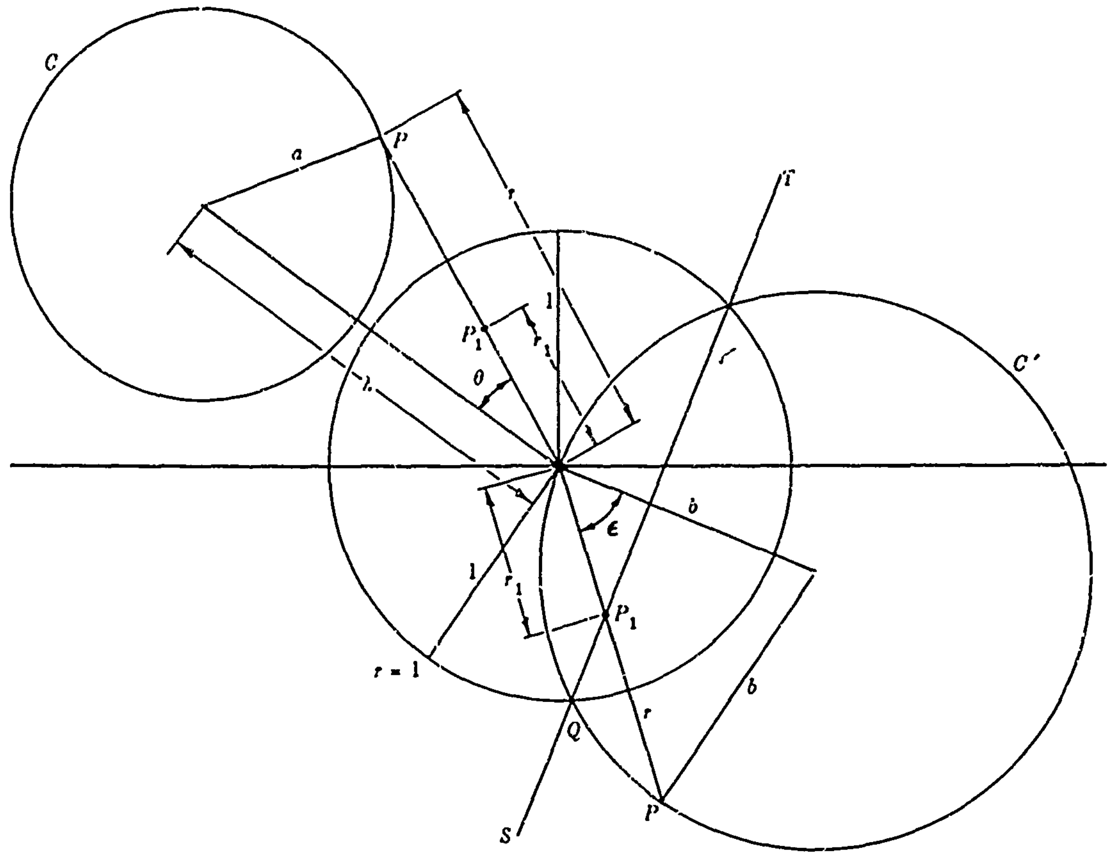


Figure 27 – Circles go into circles under the transformation $w = 1/z$.

In terms of $z = re^{i\theta}$, the two values of w are

$$w_1 = z^{1/2} = r^{1/2}\left[\cos\left(\frac{1}{2}\,\theta\right) + i\,\sin\left(\frac{1}{2}\,\theta\right)\right]$$

$$w_2 = z^{1/2} = -\,r^{1/2}\left[\cos\left(\frac{1}{2}\,\theta\right) + i\,\sin\left(\frac{1}{2}\,\theta\right)\right]$$

As z moves about on its plane, w_1 and w_2 both move about on the w-plane; their values are said to constitute different branches of the function $z^{1/2}$. The relationship is not like that of the branches of a tree, however, but rather like that of the various loops of a string tied in an open knot.

To study the situation more closely, let z start from the positive real axis and explore the z-plane without ever passing directly from the negative real axis to points below it or vice versa; it may move along curves such as ab, ac in Figure 28. The z-plane may be thought of as cut apart just below the negative real axis. Let θ be defined so that $-\pi < \theta \leq \pi$. Then w_1 will explore the right-hand half of the w-plane, including the positive half of the imaginary axis, while w_2 explores the other half of the plane. In this way w_1 maps the entire z-plane

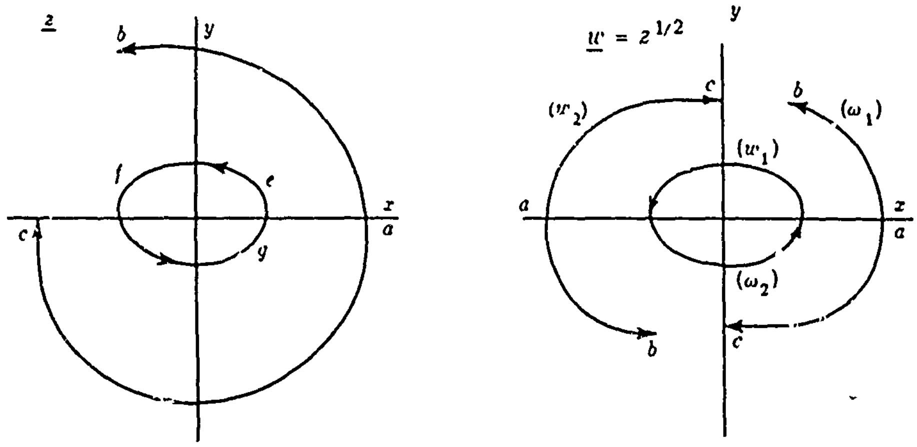


Figure 28 – The transformation $w = z^{1/2}$.

onto half of the w-plane; and w_2 maps it again onto the other half. So long as the motion of z is restricted in this manner, w_1 and w_2 behave as distinct single-valued and regular functions of z.

Yet w_1 and w_2 cannot be regarded as completely separate functions. For, if θ is allowed to vary without limit, and if z goes completely around the origin and returns to its starting point, as along curve efg (Figure 28), w_1 and w_2 will have just changed places; and, if z then explores the plane as before, w_1 and w_2 interchange roles. It is thus clear that, as z moves freely, both w_1 and w_2 move continuously and freely on the whole w-plane. Furthermore, the location of the half plane on which the entire z-plane is mapped by either value of $z^{1/2}$ can be varied at will by changing the position of the line, or curve, along which the z-plane is cut.

At $z = 0$, $w_1 = w_2 = 0$, so that the two branches come together. For this reason the point $z = 0$ is called a branch point for the function $z^{1/2}$. If z actually passes through the branch point along a continuous curve, the function $z^{1/2}$, approaching along a given branch, may be assumed to emerge without discontinuity along either branch.

The point $z = 0$ is also a singular point of a certain kind, and at this point angles are not preserved in the transformation from z to w.

The inverse transformation $z = w^2$ is single-valued. But each value of z except $z = 0$ and $z = \infty$ occurs twice among the possible values of $z = w^2$.

The function $\ln z$ is discussed in Section 40, and z^n in Section 39.

25. RELATION OF REGULAR FUNCTIONS TO TWO-DIMENSIONAL IRROTATIONAL FLOW

Consider the regular transformation [22a]

$$w = f(z) = \phi + i\,\psi$$

By differentiating Equations [22c,d] once more, it is found that

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \quad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

Thus every regular function of z furnishes at once two real solutions of the Laplace equation [7a] in two dimensions; they are obtained, respectively, from the real part and the imaginary part of the function. This principle furnishes a powerful means of discovering such solutions. Furthermore, as has been seen, the two families of curves, ϕ = constant and ψ = constant, intersect everywhere orthogonally, as illustrated in Figure 29, except perhaps where dw/dz vanishes or at a singular point.

Obviously either ϕ or ψ can be employed as the velocity potential for a type of irrotational flow.

If ϕ is the potential, the x and y components of the velocity are

$$u = -\frac{\partial \phi}{\partial x}, \quad v = -\frac{\partial \phi}{\partial y} \tag{25a,b}$$

Thus, using Equation [22c,d],

$$u = -\frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x} \tag{25c,d}$$

also. The agreement of these equations with Equations [13a,b] shows that ψ represents the stream function as previously defined.

The functions ϕ and ψ have thus all of the properties of the conjugate functions described in Section 13. The relationship is reciprocal; for, any solutions ϕ and ψ of the two-dimensional Laplace equation that satisfy Equations [22c,d] can be used to construct a regular transformation, $w = \phi + i\,\psi$. Thus conjugate functions can be defined, as an alternative, in terms of their relation to certain regular transformations of a complex variable.

Each transformation yields two conjugate types of flow, In the second type, the velocity potential ϕ' and stream function ψ' are related to ϕ and ψ by the equations $\phi' = \psi$, $\psi' = -\phi$, and the components of the velocity are

$$u' = -\frac{\partial \phi'}{\partial x} = -\frac{\partial \psi'}{\partial y} = -\frac{\partial \psi}{\partial x}, \quad v' = -\frac{\partial \phi'}{\partial y} = \frac{\partial \psi'}{\partial x} = -\frac{\partial \phi}{\partial y} \tag{25e,f}$$

Use has been made here again of the Cauchy-Riemann relations, Equations [22c,d]. This second type of flow can also be regarded as arising from the modified or conjugate transformation

$$w' = \phi' + i\,\psi' = -iw = -if(z) = \psi - i\,\phi \tag{25g}$$

Thus the conjugate flow is substituted for the original if in all formulas iw is substituted for w, since $iw' = w$.

Comparison of Equations [25e,f] and [25c,d] shows, as stated in Section 13, that the vector velocity in the second type of flow can be produced by rotating the velocity in the first

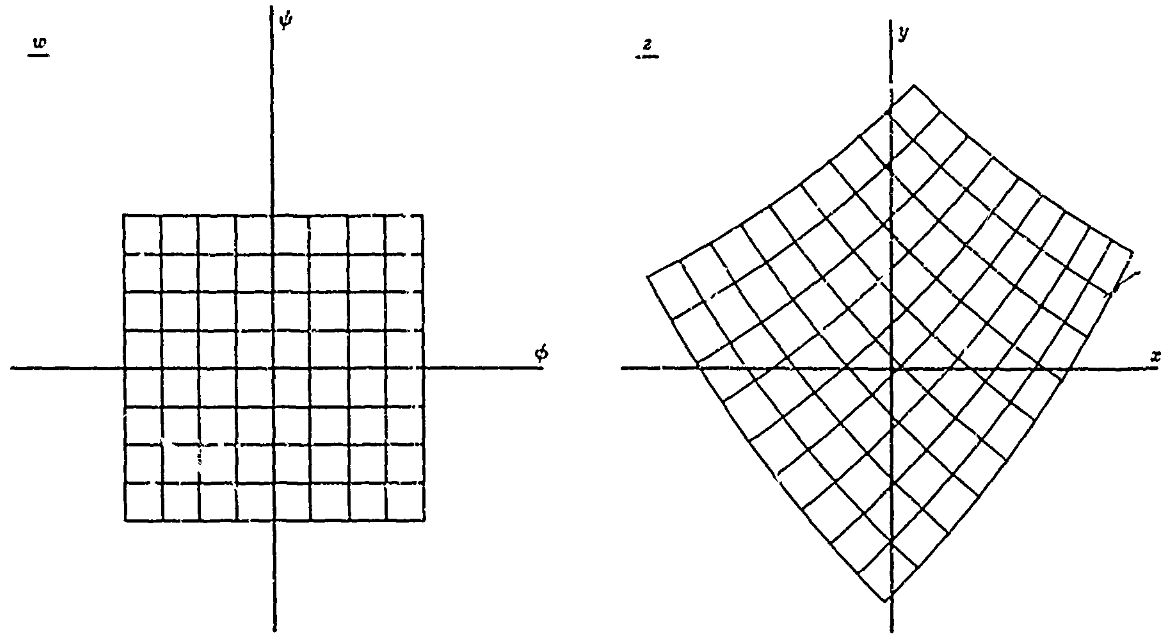


Figure 29 – Illustrating the transformation $f(z) = \phi + i\psi$ for two families of orthogonal curves ϕ = constant and ψ = constant.

type through an angle of 90 degree in the counterclockwise direction. Furthermore, the magnitude of the velocity, which has in both types the same value $q = (u^2 + v^2)^{1/2} = (u'^2 + v'^2)^{1/2}$, can be written, in view of Equation [22e];

$$q = \left|\frac{df}{dz}\right| = \left|\frac{dw}{dz}\right| \qquad [25h]$$

When ϕ is the velocity potential, it is convenient to think of w or $\phi + i\,\psi$ as a complex potential. Its derivative is related to the velocity by the equation

$$\frac{dw}{dz} = -u + iv \qquad [25i]$$

Equations [25i] and [25h] furnish usually the most convenient means of finding the velocity from Equations [22b] and [25b,c]. The points at which $dw/dz = 0$ are the stagnation points or or points of zero velocity.

At a singular point where dw/dz becomes infinite, q would be infinite. In applications of the theory, such points must be excluded. By the insertion of a boundary they may be caused to lie in a region to which the fluid does not penetrate.

So long as dw/dz is single valued, no harm results if w itself is many-valued. In that case a many-valued potential or stream function is obtained, or both.

But if, also, dw/dz is many-valued, so would be the velocity, in virtue of the relation expressed in Equation [25i]. A many-valued velocity, however, is physically impossible. In such cases the z-plane must be cut or divided by a curve, representing a physical boundary,.

in such a way that, as z varies freely but without touching this curve, w varies continuously along one branch of its values without the occurrence of ambiguities as to the value of dw/dz. Examples of this procedure occur in the next chapter; Sections 40 and 61 may be mentioned.

Many transformations are most conveniently defined in the inverse form as $z = F(w)$. Upon separating real and imaginary parts, equations of the form $x = F_1(\phi, \psi)$, $y = F_2(\phi, \psi)$ are obtained. From these equations the equipotential curves and streamlines, defined by constant values of ϕ and ψ, may be traced.

Finally, the physical significance of certain constants that may be introduced into a transformation should be noted.

Consider, in the first place, the effect of replacing

$$w = f(z) \qquad [25j]$$

by

$$w = f(Az + B), \text{ or } w = f\left[k(z-h)\, e^{-i\alpha}\right] \qquad [25k,l]$$

where $k = |A|$, $\alpha = -amp\ A$, so that $A = ke^{-i\alpha}$, and $h = -Be^{i\alpha}/k$. The value of w that is associated with any given value z_1 of z by the equation $w = f(z)$ is assigned by (25l) to a value z_2 such that

$$k(z_2 - h)\, e^{-i\alpha} = z_1, \text{ or } z_2 = \frac{z_1}{k} e^{i\alpha} + h$$

Thus the vector representing z_2 is obtained from that for z_1 by changing its magnitude in the ratio $1/k$ and also rotating it through the angle α, and then adding the vector representing h. The resulting change in the plot of w on the z-plane can thus be described by supposing the plot to be changed in scale in the ratio 1/k and also rotated counterclockwise through an angle α, without moving the origin, and then to be given the translation represented by the real or complex number h. The entire flow is thus rotated and displaced on the z-plane in the manner described. This constitutes an important means by which the solutions of hydrodynamical problems can be modified to suit new conditions.

The changes produced in ϕ and ψ regarded as functions of x and y by the rotation and displacement of the plot are the same as would result from an opposite rotation and displacement of the x and y axes and thus possess in themselves no novelty. The change of scale, however, is less familiar. It leads to the useful rule that all functions or expressions resulting from a transformation $w = f(z)$ may be generalized by replacing everywhere z by kz, or x, y by kx, ky, where k is any real number.

In the second place, consider the effect of replacing $w = f(z)$ by

$$Cw + D = f(z) \qquad [25m]$$

Writing $C = C_1 + i\,C_2$ where C_1 and C_2 are real, $Cw = C(\phi + i\,\psi) = C_1\phi - C_2\psi + i(C_2\phi + C_1\psi)$. Thus C changes the potential and the stream function from ϕ and ψ to

$$\phi' = C_1\,\phi - C_2\,\psi, \quad \psi' = C_1\psi + C_2\,\phi$$

Here, in the terms containing C_2, ψ may be regarded as a second possible potential and $-\phi$ as the corresponding stream function. It is already known, however, that a new potential and the associated stream function can be constructed by making a linear combination of other potentials and the same combination of the associated stream functions.

The addition of D to Cw then merely adds constants to ϕ' and ψ', which, as hydrodynamical quantities, contain arbitrary constants in any case.

In view of all these results, it is often convenient to study a transformation in skeleton form, with the omission of constants such as A, B, C, D. The equations thus obtained may not be dimensionally balanced, from the physical standpoint. The results can then easily be generalized as desired by adding constants to ϕ and ψ, or by multiplying both of them in all equations by the same real constant, or by making suitable combinations of these functions, or by changing axes on the z-plane, or, finally, by multiplying z, x, and y in all equations by the same real number. In this way, also the dimensional balance can be restored if desired.

In practical problems a boundary condition is usually specified. If the fluid is confined by fixed bounding surfaces, the streamlines must be tangential to these surfaces, and over each of them ψ must have a fixed value. The mathematical problem is then to find a transformation $w = f(z)$ such that the curves representing these surfaces on the z-plane transform on the w-plane into straight lines parallel to the ϕ axis, along each of which ψ has a constant value.

No practical general method of discovering the necessary transformation is known. It can sometimes be found by means of the Schwarz-Christoffel transformation, which will be described presently. Many types of flow have been discovered by assuming some transformation and then investigating the flow that it represents.

26. THE TRANSFORMATION OF IRROTATIONAL MOTIONS

The solution of a new problem can sometimes be obtained by transforming the known solution of an old one. Thus, let $w = \phi + i\,\psi = f(z)$ be the complex potential for a known problem; and let z be connected with a new variable Z by the transformation

$$z = x + iy = F(Z),\ Z = X + iY$$

The result is equivalent to a single transformation from Z to w:

$$w = f[z(Z)] = g(Z)$$

Hence, when $\phi(x,y)$ and $\psi(x,y)$ have been expressed in terms of X and Y, they may be taken as the potential and the stream function for a new motion described in terms of X and Y. The original boundaries on the z-plane become transformed into boundaries of a different shape on the Z-plane; and the curves, ϕ = constant and ψ = constant, transform into curves for the same constant values of ϕ and ψ on the Z-plane. Thus the known flow described in terms of z is transformed into another type of flow satisfying different boundary conditions.

An alternative mathematical statement is the following. Let $\phi(x,y)$, $\psi(x,y)$ be a known pair of conjugate functions, and let $x(X,Y)$, $y(X,Y)$ be any other pair of conjugate functions in terms of the variables X and Y. Then a new pair of conjugate functions in terms of X and Y can be obtained by substituting in $\phi(x,y)$ and $\psi(x,y)$ the expressions for x and y in terms of X and Y. They may be written $\phi[x(X,Y), y(X,Y)]$, $\psi[x(X,Y), y(X,Y)]$

Any boundary that is a streamline on the z-plane remains a streamline on the Z-plane. Sources and sinks also remain sources and sinks of the same strength; and the circulation around any closed curve retains the same value around the transformed curve. For, the volume of fluid emitted from a line source, per second and per unit length, is represented by the decrease in ψ as the source is encircled once in the positive direction, according to a principle stated in Section 40, whereas the circulation around a closed curve is similarly represented by the decrease in ϕ as the curve is traversed in the positive direction, and these changes in ϕ and ψ are invariant under the transformation.

27. THE LAURENT SERIES

Many series of positive powers are limited in their range of convergence. For example,

$$\frac{1}{1-z} = 1 + z + z^2 + \dots .$$

converges only within the unit circle defined by $|z| = 1$. On the other hand, negative powers such as $1/z$ or $1/z^2$ are regular functions of z except only at $z = 0$. These observations suggest that series containing both positive and negative powers might be useful.

In books on functions of a complex variable it is shown that, if $f(z)$ is regular at all points near a given point $z = c$, it can be expanded in a series of the form

$$f(z) = \dots b_2(z-c)^{-2} + b_1(z-c)^{-1} + a_0 + a_1(z-c) + a_2(z-c)^2 \dots .$$

where the a's and b's are constants and all positive and negative powers of $(z-c)$ may occur. This is called a Laurent Series. It converges at any $z \neq c$ throughout the interior of a circle drawn about c as center and passing through the singularity nearest to c; if $f(z)$ has no singularity except perhaps at c itself, the series converges for all $z = c$.

If the series contains negative powers of unlimited order with nonvanishing coefficients, $f(z)$ has an essential singularity at $z = c$; if the series begins with a term containing a definite negative power, namely, $b_m(z-c)^{-m}$, $f(z)$ has a pole of order m at $z = c$; if $m = 1$, the pole is called simple. If $f(z)$ is regular also at c, the negative powers disappear and the series becomes a Taylor series, converging also at $z = c$.

The series exists also if $f(z)$ is assumed to be regular merely outside of a given circle centered at c, or between two such circles. Then the series converges at least at all points outside of the given circle, or between the two circles, respectively.

In any case, if a Laurent Series or Taylor Series representing a function $w(z)$ converges for all large z, then it can be shown that either values of $|w|$ exceeding all limits occur when z goes to infinity in certain directions, or else the series contains no positive powers of z.

28. COMPLEX INTEGRATION

An integral with respect to the complex variable $z = x + iy$ is defined in the same way as with real variables, but it has some novel properties.

The indefinite integral of $f(z)$ or $\int f(z)\,dz$ is a function $F(z)$ of z whose derivative is $f(z)$, as with real variables. If $F(z)$ is many-valued, care must be taken to select a branch of this function that varies continuously with z.

In defining the definite integral, it is necessary to specify, in addition to the limits, a definite path of integration connecting them. This may be indicated by adding to the integral sign a symbol designating the path. For example, the integral of $f(z)$ along the path APB in Figure 30 is

$$\int_{(APB)} f(z)\,dz = \lim_{\Delta z \to 0} \Sigma f(z)\,\Delta z$$

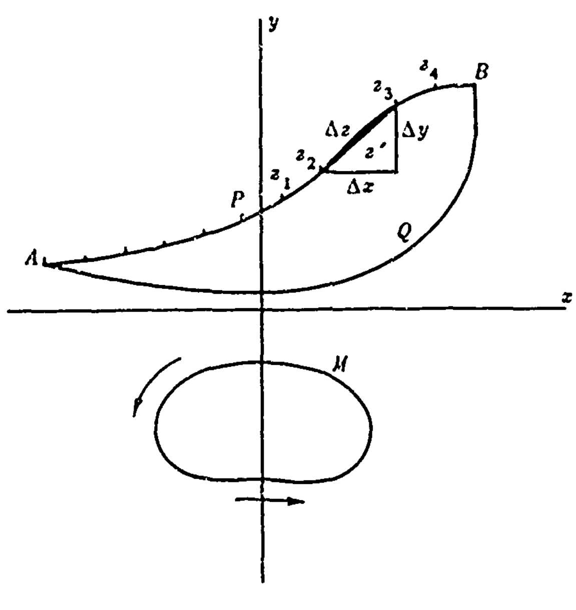


Figure 30 – Illustrating the definition of a complex integral.

Here the sum on the right is formed as follows. Choose a large number of points scattered along the curve, and let Δz stand for the difference in the values of z at any two successive points; thus, in Figure 30, one $\Delta z = z_3 - z_2$, the next $\Delta z = z_4 - z_3$, and so on. Multiply each Δz by the value of $f(z)$ at any point on the corresponding segment of the curve; for example, if $\Delta z = z_3 - z_2$, $f(z)\,\Delta z$ may stand for $f(z')\,(z_3 - z_2)$, where z' is the point shown in Figure 30. All the products thus obtained are to be added, and the limit of this sum is to be taken as the number of points is increased indefinitely in such manner that all of the differences Δz approach zero.

The value of such an integral is usually a complex number. It can also be written in terms of two real line integrals in which the variables of integration are x and y. Thus, if $f(z) = \phi(x,y) + i\,\psi(x,y)$ where ϕ and ψ are real functions, since $dz = dx + i\,dy$,

$$\int f(z)\, dz = \int (\phi\, dx - \psi\, dy) + i \int (\psi\, dx + \phi\, dy) \qquad [28a]$$

Here dx and dy may be interpreted as components of successive elements dz, and values of ϕ and ψ are to be taken at points lying on the corresponding elementary segments of the path. Negative values of dx and dy may occur as well as positive values.

The integral of $f(z)$ along a closed curve is often denoted by $\oint f(z)\, dz$. This symbol will be understood to imply that the curve is traversed in the positive or counterclockwise direction, that is, in such a direction that its interior lies on the left.

29. THE CAUCHY INTEGRAL THEOREM

As in the case of integrals with respect to real variables, reversing the direction of integration along the path reverses the sign of the value of $\int f(z)\, dz$. But integrals of $f(z)$ along different end points, such as APB and AQB in Figure 30, may or may not be equal; and $\oint f(z)\, dz$ taken around a closed path or contour, such as M in Figure 30, may or may not vanish. If an integral around a contour does not vanish, its sign is changed if the direction of integration around the contour is reversed.

The following important theorems can, however, be proved. The first two taken together are known as Cauchy's integral theorem.

(a) If $f(z)$ is regular at all points both inside of and on a closed contour, then around the contour $\oint f(z)\, dz = 0$.

(b) If $f(z)$ is regular at all points between and on two paths joining two end points P and P', then $\int_P^{P'} f(z)\, dz$ has the same value along both paths.

(c) If $f(z)$ is regular at all points between and on two closed contours of which one encloses the other, then $\oint f(z)\, dz$ has the same value around both contours.

In all three cases, it is also sufficient if $f(z)$, instead of being actually regular on the contour or path itself, is merely continuous from the contour or path into the region in which it is required to be regular.

Cauchy's second proof of (a) is instructive enough to be repeated here. It is open to a certain logical objection, however; a more satisfactory proof can be found in books on functions of a complex variable (for example, E.T. Copson [13]).

Let the first of the Cauchy-Riemann equations or [22c] be integrated with respect to x and y over the area on the z-plane enclosed within the contour, giving

$$\iint \frac{\partial \phi}{\partial x}\, dx\, dy = \iint \frac{\partial \psi}{\partial y}\, dx\, dy$$

Now

$$\iint \frac{\partial \phi}{\partial x} dx\, dy = \int dy \int \frac{\partial \phi}{\partial x} dx = \int (\phi_2 - \phi_1)\, dy$$

provided $\partial\phi/\partial x$ is continuous in x. Here ϕ_2 and ϕ_1 denote values at opposite ends of the range for x, for any given value of y, as illustrated in Figure 31.

The integral in y is then to be carried out between the extreme limits for y, and dy is is here understood to be positive. This integral can also be written

$$\int (\phi_2 - \phi_1)\, dy = \oint \phi dy$$

where $\oint$ denotes as usual the integral taken around the contour in the counterclockwise direction. For $\phi_2\, dy$ equals the corresponding ϕdy in the contour integration, whereas $\phi_1\, dy = -\phi\, dy$ since all dy's are negative along the left-hand side of the contour. Hence

$$\iint \frac{\partial \phi}{\partial x}\, dx\, dy = \oint \phi\, dy$$

Similarly,

$$\iint \frac{\partial \psi}{\partial y} dx\, dy = \int dx \int (\psi_2 - \psi_1)\, dy = -\oint \psi\, dx$$

the sign is negative here because it is at point number 2 that dx has opposite signs in the two integrations.
Hence

$$\oint dy = -\oint \psi\, dx$$

Similarly, by integrating [22d]

$$-\oint \phi\, dx = -\oint \psi\, dy$$

From these two equations it is obvious that the right-hand member of Equation [28a] vanishes. Hence $\oint f(z)\, dz = 0$.

Theorems (b) and (c) are corollaries of (a).

To deduce (b), let APB and AQB denote two paths of the kind specified in (b). Then APBQA is a closed path to which (a) applies, so that

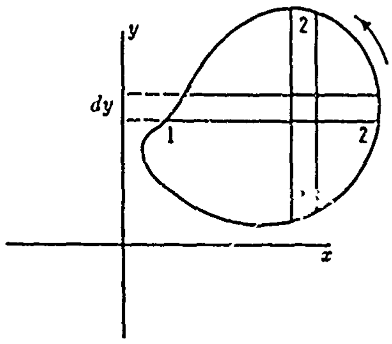


Figure 31 – A closed path of integration

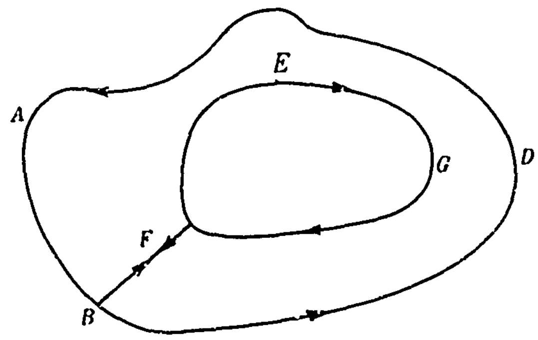


Figure 32 – Two alternative closed paths of integration.

$$\int_A^B (P)\, f(z)\, dz + \int_B^A (Q)\, f(z)\, dz = 0$$

Here (P) and (Q) are inserted to specify that the paths of integration pass respectively through the points P and Q, But

$$\int_B^A (Q)\, f(z)\, dz = - \int_A^B (Q)\, f(z)\, dz$$

hence

$$\int_A^B (P)\, f(z)\, dz = \int_A^B (Q)\, f(z)\, dz$$

To obtain theorem (c), connect any two contours ABDA, EFGE, by a cross-path BF, as in Figure 32. Then the path BDABFEGFB is a closed contour around which, under the conditions assumed in theorem (c), $\oint f(z)\, dz = 0$. But the path BF is traversed twice, in opposite directions, and hence its net contribution to the integral vanishes. The contributions made by the original contours are thus equal and opposite. But the contour EFGE was traversed with its interior on the right, or in the negative direction; if traversed positively, its contribution to the integral is reversed in sign. Hence $\oint f(z)\, dz$ has the same value around the two original contours.

These theorems are of enormous value in the evaluation of complex integrals. The evaluation can often be rendered very easy by deforming the path into a suitable shape; the path can be deformed at will so long as it is not deformed past any point at which $f(z)$ ceases to be regular.

30. SINGULAR POINTS AND RESIDUES

An important case in applications of the theory is that in which $f(z)$ is regular throughout a certain region S except at one or more internal points. These excluded points may be singular points, or they may be points at which nothing is known or assumed about the function.

Suppose that S contains one excluded point. Then $\oint f(z)\, dz$ has the same value for all closed curves lying in S which encircle this point once. This follows from theorem (c) in Section 29, in view of the fact that no excluded point occurs either between or on the two curves. The number

$$\frac{1}{2\pi i} \oint f(z)\, dz$$

is called the *residue* of the function $f(z)$ at the excluded point.

If more than one excluded point occurs in S, the value of $\oint f(z) dz$ around a curve encircling any finite number of them is $2\pi i$ times the sum of the residues of (z) at the encircled points. This is proved by deforming the original curve until it consists of separate curves encircling one singular point each and connected by paths that are traversed twice, as illustrated in Figure 33, where Q and R represent two excluded points and the outer curve is the original one. The connecting paths contribute nothing to $\oint f(z)\, dz$ taken around the combined curve.

As an example, consider

$$f(z) = \frac{k}{(z-a)^n}$$

where n is a positive integer and a and k are constants. This function has one singularity, at $z=a$. Let the path of integration be a circle of radius R about $z=a$ as center. Then, for values of z on the circle, $|z-a| = R$, and, if θ is the amplitude of $z-a$,

$$z-a = Re^{i\theta},\ dz = iRe^{i\theta}\, d\theta$$

since R is constant along the circle. Thus

$$\oint \frac{k\,dz}{(z-a)^n} = k\int \frac{iRe^{i\theta}\, d\theta}{(Re^{i\theta})^n} = i\,kR^{1-n} \int_0^{2\pi} e^{i(1-n)\theta}\, d\theta$$

If $n > 1$,

$$\oint \frac{k\,dz}{(z-a)^n} = ikR^{1-n}\ \frac{e^{i(1-n)\theta}}{i(1-n)} \Bigg|_{\theta=0}^{\theta=2\pi} = 0$$

since, for integral n,

$$e^{i(1-n)2\pi} = \cos[(1-n)2\pi] = 1$$

But if $n = 1$ the integral becomes

$$\oint \frac{k\,dz}{z-a} = ik \int_0^{2\pi} d\theta = 2\pi ik \qquad [30a]$$

Thus the residue of $k(z-a)^{-n}$ is 0 for $n > 1$ but equal to k for $n = 1$. Often $f(z)$ can be written in the form

$$f(z) = \frac{g(z)}{(z-a)^m}$$

where m is a positive integer and the function $g(z)$ is regular both at $z = a$ and in its neighborhood. Then $g(z)$ can be expanded in a Taylor series near $z = a$:

$$g(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots$$

By substituting this series for $g(z)$ and using the results just obtained, it is seen that the residue of $f(z)$ at $z = a$ is a_{m-1} or the coefficient of the power $(z-a)^{m-1}$ in the series. Or, the residue of $f(z)$ at $z = a$ also equals $g^{(n)}(a)/n!$ where $g^{(n)}(a)$ denotes the value of the n^{th} derivative of g at $z = a$.

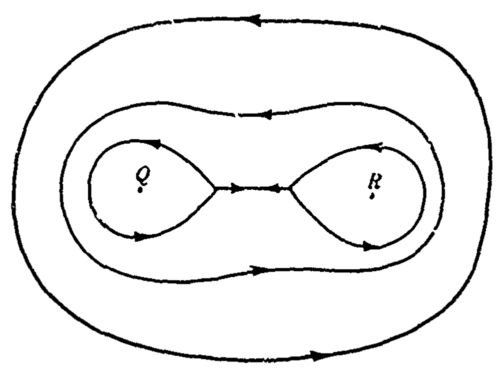


Figure 33 – Integration around two singular points Q, R.

31. THE SCHWARZ-CHRISTOFFEL TRANSFORMATION

This transformation is useful in two-dimensional hydrodynamical problems that involve boundaries in the form of flat surfaces, so that their trace on the xy-plane is a polygon. It may be an ordinary finite closed polygon, such as $A_1\,A_2\,A_3\,A_4\,A_5$ in Figure 34, or the boundary on the xy-plane may consist of one or more broken lines each of which extends to infinity in at least one direction. Boundaries of the latter sort can be formed out of a finite polygon by allowing one or more vertices to recede to infinity and perhaps to spread out there; they are often regarded as closed polygons with vertices at infinity.

The Schwarz-Christoffel transformation maps the sides of such a polygon onto the real axis in another complex plane, and maps the interior of the polygon into the upper half of this plane. If the polygon has vertices at infinity, the space on either side of it may be defined as the interior.

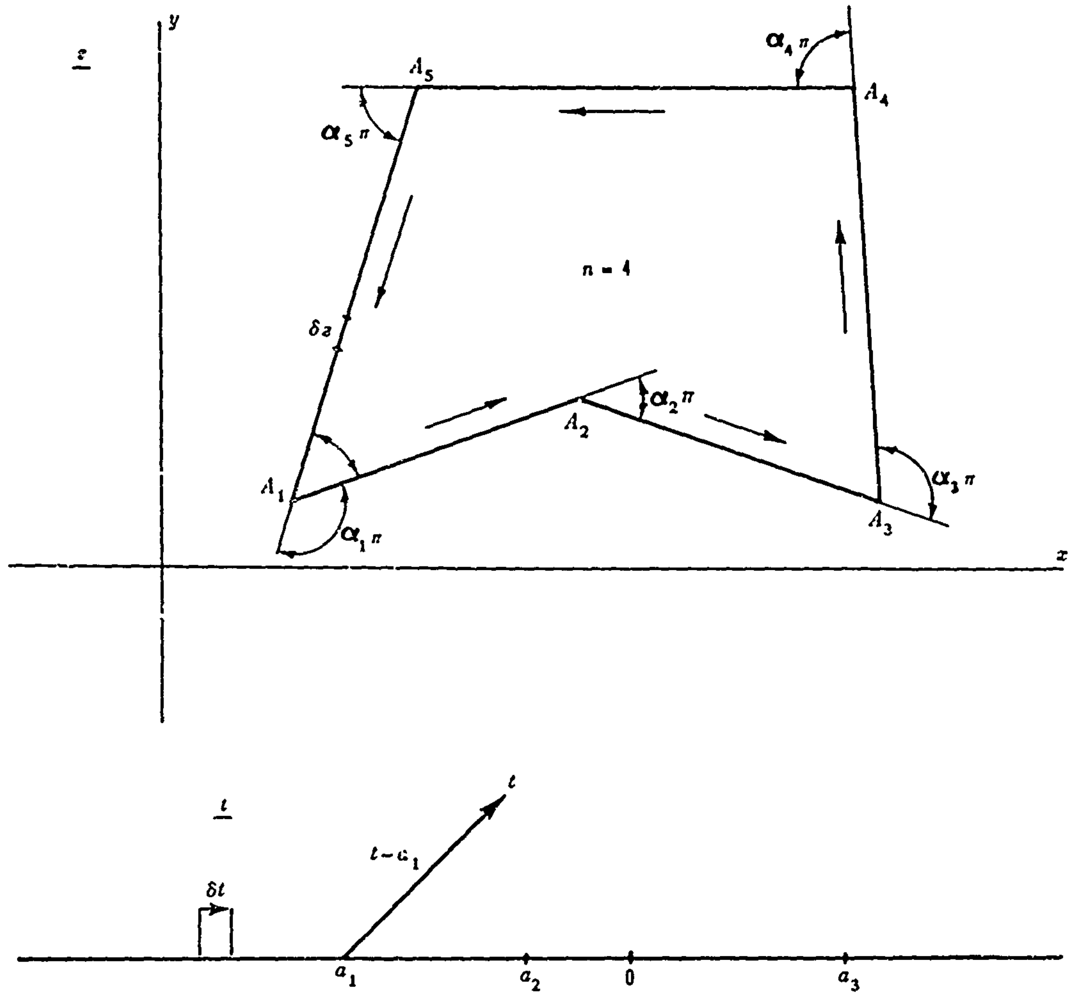


Figure 34 – Illustrating the Schwarz-Christoffel transformation.

Let the polygon be drawn on the z-plane and mapped onto the plane of the variable t, which is also shown in Figure 34, t being complex.

The appropriate transformation is most simply stated in terms of the inverse derivative, thus:

$$\frac{dz}{dt} = K(t-a_1)^{-\alpha_1}(t-a_2)^{-\alpha_2}\ldots(t-a_n)^{-\alpha_n} \qquad [31a]$$

Here K is a constant, real or complex; $a_1, a_2 \ldots a_n$ are n real numbers in ascending order of magnitude; and $\alpha_1, \alpha_2 \ldots \alpha_n$ are another set of n real numbers.

The powers that occur in [31a] must be made single-valued by a suitable convention concerning the amplitudes. Let a denote any one of the constants $a_1 \ldots a_n$, and α the corresponding $\alpha_1 \ldots \alpha_n$. When t is real and $t < a$, $t - a$ is a negative real number. As t explores the upper half plane and comes down to the real axis where $t > a$, as illustrated in Figure 35, the amplitude θ of $t - a$ decreases by π. Let θ be so chosen that

$$-\frac{\pi}{2} < \theta \leq \frac{3}{2}\pi$$

Then, for the values of t under consideration, θ varies continuously between 0 and π, and $(t-a)^{-\alpha\theta}$ is a continuous function and is differentiable by the ordinary rule. Actually, t may be allowed to go anywhere except to $t = a$, but it must not cross the vertical line extending downward from $t = a$.

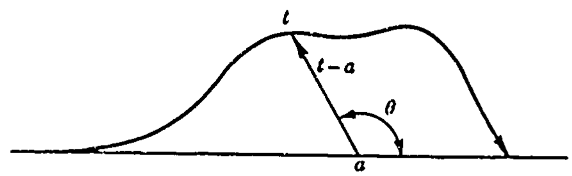


Figure 35 – Path past a singular point.

Let all of the points $t = a_1, a_2 \ldots a_n$ be treated in this manner. Then dz/dt and the function $z(t)$ obtained by integrating dz/dt will be regular functions of t above and on the real axis except at the points $t = a_1, a_2 \ldots a_n$.

Finite Polygons. Let $\alpha_1, \alpha_2 \ldots \alpha_n$ be such that

$$-1 \leqq \alpha_j < 1, \quad j = 1, 2 \ldots n$$

$$1 < \alpha_1 + \alpha_2 + \ldots \alpha_n \leqq 2$$

The meaning of the first statement is that all of the α's lie within the limits specified.

Under these conditions, the transformation defined by Equation [31a] transforms the real axis of t into a closed polygon on the z-plane. To show this, the equation must be integrated along the real axis.

As t advances a distance δt along its real axis, the z-point on the z-plane undergoes a displacement

$$\delta z = \frac{dz}{dt}\,\delta t$$

Since δt as a vector is directed toward $t = t + \infty$, the direction of δz will make with the real axis on the z-plane an angle equal to the amplitude of dz/dt. This amplitude is in turn the sum of the amplitudes of the various factors in the right-hand member of Equation [31a].

So long as t is to the left of a_1, the amplitudes of all factors such as $(t-a)^{-\alpha}$ remain constant, and so does amp (dz/dt). Thus, as t moves from $-\infty$ up to a_1, z moves along a straight line, as illustrated, for example, by $A_5 A_1$ in Figure 34.

The total complex length of this line is

$$K \int_{-\infty}^{a_1} (t-a_1)^{-\alpha_1} (t-a_2)^{-\alpha_2} \ldots (t-a_n)^{-\alpha_n} dt$$

The integral is an improper one, but under the conditions assumed it converges at both limits and has a finite value.

For, in the first place, over a short range of t from some value t_1 up to a_1, variation of the remaining factors can be neglected and the corresponding part of the integral is nearly proportional to

$$\int_{t_1}^{a_1} (t-a_1)^{-\alpha_1} dt = (1-\alpha_1)^{-1} (t-a_1)^{1-\alpha_1} \Big/_{t_1}^{a_1} \qquad [31b]$$

Since by assumption $\alpha_1 < 1$, this integral is finite.

In the second place, for large negative t the constants $a_1, a_2 \ldots$ may all be dropped in comparison with t. Thus the integral toward $t = -\infty$ reduces approximately to

$$K \int_{-\infty} (t)^{-(\alpha_1 + \alpha_2 + \ldots \alpha_n)} dt = \frac{A}{1-(\alpha_1+\alpha_2+\ldots \alpha_n)} t^{1-(\alpha_1+\alpha_2+\ldots \alpha_n)} \Big/_{t=-\infty} \qquad [31c]$$

which is finite since the sum of the α's has been assumed to exceed 1.

As t increases past a_1, the amplitude of $t-a$ decreases from π to 0, as is clear from Figure 35. Hence the amplitude of $(t-a)^{-\alpha_1}$ increases from $-\alpha_1 \pi$ to 0, and the amplitude of dz/dt likewise increases by $\alpha_1 \pi$. Thus from $t = a_1$ to $t = a_2$, z travels along another straight line making an exterior angle $\alpha_1 \pi$ with the first line. This line, too, is of finite length, as illustrated by $A_1 A_2$ in Figure 34.

It cannot be concluded immediately, however, that these two lines join at A_1. For it may not be possible actually to integrate past the point $t = a_1$, at which dz/dt is infinite if α_1 is positive. To avoid this difficulty, we adopt the standard device of letting t pass above a_1 along a small semicircle centered at a_1, as illustrated in Figure 36. As t traverses this semicircle, z cuts across from one straight line to the other, along a curve such as that drawn near A_1 in Figure 34. The change in z along this curve is given by the integral of dz/dt along the semicircle.

In terms of polar coordinates, as illustrated in Figure 36, on the semicircle

$$t-a_1 = r_1 e^{i\theta_1}, \quad dt = ir_1 e^{i\theta_1} d\theta_1$$

since r_1 is constant. For an approximate estimate, all other factors in dz/dt can be treated as constants; let their product, multiplied by K, be denoted by Q. Then the change in z as t goes around the semicircle is, from [31a],

$$\Delta z = \int \frac{dz}{dt}\, dt = i Q\, r_1^{\,1-\alpha_1} \int_{\pi}^{0} e^{i(1-\alpha_1)\theta_1}\, d\theta_1$$

$$= \frac{Q}{1-\alpha_1}\, r_1^{\,1-\alpha_1} \left(1 - e^{i(1-\alpha_1)\pi}\right)$$

Since $\alpha_1 < 1$, the exponent of r_1 is positive. Hence, as the semicircle is shrunk down onto $t = a_1$ and $r_1 \to 0$, $\Delta z \to 0$. On the z-plane, therefore, the two lines must meet at a point.

By proceeding in this manner it can be shown that, as t traverses its real axis from $-\infty$ to $+\infty$, z moves along a broken line with corners corresponding to $t = a_1, a_2 \ldots a_n$, at which exterior angles $\alpha_1 \pi, \alpha_2 \pi \ldots \alpha_n \pi$ occur. To form a finite polygon, the ends of this broken line must coincide.

Now the distance between the ends is equal to $\int_{-\infty}^{\infty} (dz/dt)\, dt$ along the entire real axis, calculated with avoidance of all the singular points in the manner just described. The value of the integral can be found more easily by the following indirect method.

Let t trace the following contour, as illustrated in Figure 37. Beginning at a point $t = -R$ where R is a large positive real number, let t trace the real axis to the point $t = R$, except that it goes round above the points $a_1, a_2 \ldots a_n$ along small semicircles. Let t then return to its starting-point along a large semicircle of radius R. On this contour and everywhere inside it, dz/dt is differentiable. Hence, by the Cauchy integral theorem, $\int (dz/dt)\, dt$ around the contour vanishes. But it can be shown that the contribution of the large semicircle decreases to zero as $R \to \infty$. For, on this semicircle, the absolute value of t equals R and is so large that $a_1, a_2 \ldots a_n$ are relatively negligible and may be omitted. Let

$$t = Re^{i\theta}, \; dt = i R e^{i\theta}\, d\theta$$

and write

$$\alpha_1 + \alpha_2 + \ldots \alpha_n = \Sigma \alpha$$

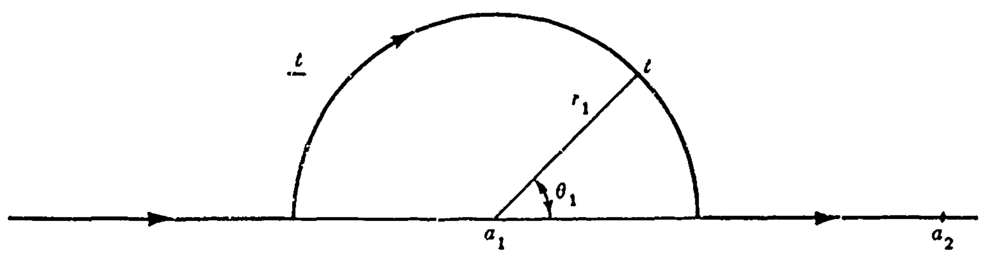


Figure 36 – Semicircular path past a singular point.

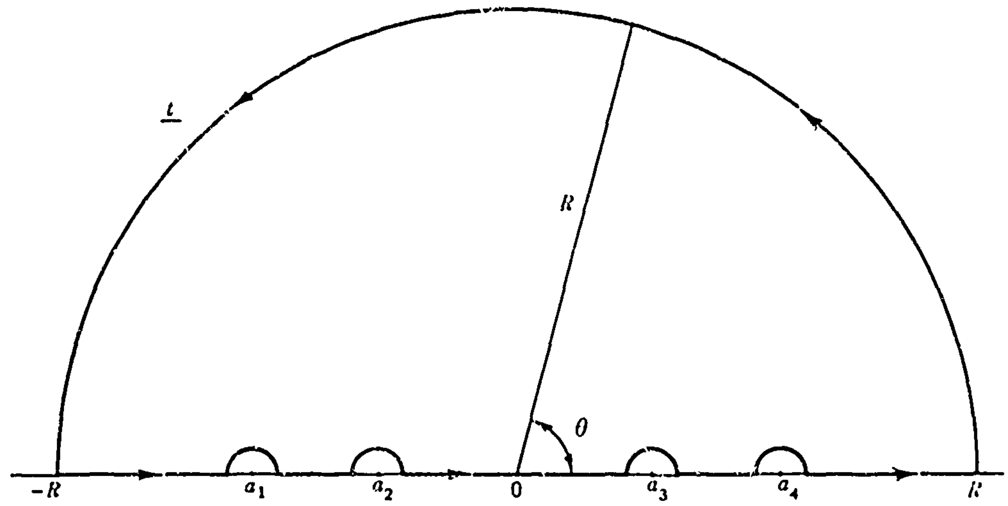


Figure 37 – A large semicircular path.

Then, from [31a], around the large semicircle, approximately,

$$\int \frac{dz}{dt}\,dt = K\int t^{-\Sigma\alpha}\,dt = i\,KR^{1-\Sigma\alpha}\int_0^{\pi} e^{i(1-\Sigma\alpha)\theta}\,d\theta$$

$$= \frac{K}{1-\Sigma\alpha}\,R^{1-\Sigma\alpha}\left(e^{i(1-\Sigma\alpha)\pi}-1\right) \qquad [31d]$$

Since by assumption $\Sigma\,\alpha > 1$, this last expression goes to 0 as $R \to \infty$.

The remainder of the contour integral, therefore, must likewise become zero as $R \to \infty$. But if the small semicircles are allowed to shrink down onto their centers, the remainder becomes in the limit the desired integral. Hence it must be that this latter integral itself equals zero. It follows that $\Delta z = 0$ between the ends of the broken line, so that they join and complete a finite polygon, at A_5 in Figure 34.

The last segment of the broken line makes an angle $(\Sigma\alpha)\,\pi$ with the first, or the first makes an angle

$$\alpha_{n+1}\,\pi = (2-\Sigma\alpha)\,\pi = [2-(\alpha_1+\alpha_2\,\cdots\,\alpha_n)]\,\pi$$

with the last; in Figure 34 this is the exterior angle at A_5. The number of actual vertices then depends upon the value of $\Sigma\,\alpha$.

If $\Sigma\,\alpha = \alpha_1 + \alpha_2 \cdots \alpha_n < 2$, the polygon has an actual vertex corresponding to $t = \infty$, with an exterior angle $\alpha_{n+1}\pi$. The two adjacent sides $A_5\,A_1$ and $A_4\,A_5$ in Figure 34, have lengths

$$\int_{-\infty}^{a_1} [\text{real}]\,\frac{dz}{dt}\,dt, \quad \int_{a_n}^{\infty} [\text{real}]\,\frac{dz}{dt}\,dt$$

where [real] indicates that the integral is taken along the real axis. Thus, in general, n factors in dz/dt produce a polygon with $n+1$ vertices and sides, and with $t = \infty$ at one vertex.

If $\alpha_1 + \alpha_2 + \ldots \alpha_n = 2$, however, $\alpha_{n+1} = 0$, so that the first and last segments of the broken line coalesce into a single straight line. In this case the polygon has only n actual vertices and sides, and $t = \infty$ occurs somewhere on one side; the total length of this side is

$$\int_{-\infty}^{a_1} [\text{real}] \frac{dz}{dt}\, dt + \int_{a_n}^{\infty} [\text{real}] \frac{dz}{dt}\, dt$$

This case may be regarded as a degenerate one in which the exterior angle at the $(n+1)$ st vertex is zero.

Thus it has been shown that the real axis of t is transformed into a finite closed polygon. It remains then to show that the arbitrary constants in the transformation can be chosen so as to fit an arbitrarily chosen polygon on the z-plane.

The general expression for z will be

$$z = K \int (t-a_1)^{-\alpha_1} (t-a_2)^{-\alpha_2} \ldots (t-a_n)^{-\alpha_n}\, dt + L$$

Now changing the integration constant L merely translates the polygon on the z-plane; changing $|K|$ stretches all of its sides in a certain ratio, and changing *amp* K rotates it about the point $z = L$. By adjusting K and L, therefore, one side of the transformed polygon can always be made to coincide with one side of the given polygon. The two polygons will then coincide completely provided they have the same shape. The necessary similarity can be secured in either of two alternative ways.

1. For a polygon of m sides, $m-1$ factors may be employed in the expression for dz/dt, with $\alpha_1, \alpha_2 \ldots \alpha_{m-1}$ made equal to $m-1$ external angles of the given polygon taken in order, each divided by π. The external angles then come out correct. For the lengths of the sides, m integrals are obtained, two of them extending to $t = \pm \infty$. Elimination of the factor K from these integrals leaves $m-1$ ratios between them. By a suitable choice of $a_1, a_2 \ldots a_{m-1}$, these ratios can be made equal to the $m-1$ ratios of the lengths of the sides of the given polygon to the length of a chosen side. These latter ratios cannot all be independent, however; for the last two sides, whose directions are already fixed, will automatically come into the correct ratio when the other ratios have been adjusted. In Figure 34, for example, $A_3 A_4$ and $A_4 A_5$ are fixed in length when their directions have been assigned and when the sides $A_1 A_2$ and $A_2 A_3$ have been constructed in the proper ratio to $A_5 A_1$. Hence two of the a's can be chosen arbitrarily, the remaining a's being then chosen so as to give correct values to $m-3$ of the ratios of the sides. In practice, it is usually most convenient to determine K and L by substituting the values of z at two corners.

2. As an alternative, m factors may be employed in the expression for dz/dt, with α 's representing all of the external angles. In this case only one integral to infinity is obtained, representing the length of one of the m sides, at some point of which $t = \infty$ occurs. The $a's$ are again subject to $m - 3$ conditions, but here their number is m. Hence in this case three of the $a's$ can be chosen arbitrarily.

Infinite Polygons. A corner of the polygon can be displaced to infinity in either of two ways.

1. If $-1 \leq \alpha_1 + \alpha_2 \ldots \alpha_n \leq 1$, the integrals to $t = +\infty$ and from $t = -\infty$ no longer converge, as is illustrated by Equation [31c]. Thus the broken line extends to infinity at both ends. The integral along the semicircle at infinity, in Equation [31d], also no longer vanishes.

If $\alpha_1 + \alpha_2 + \ldots \alpha_n = 1$, $\alpha_{n+1} = 2 - (\alpha_1 + \ldots \alpha_n) = 1$ also, and the first and last segments of the broken line, on which $t < a$, or $t > a_n$, respectively, differ in direction by $\alpha_{n+1}\,\pi = \pi$ and so are geometrically parallel. In this case [31a] can also be written

$$\frac{dz}{dt} = -K\,(a_1 - t)^{-\alpha_1} \ldots (a_n - t)^{-\alpha_n}$$

Thus on the first and last segments dz/dt has opposite signs, so that these segments are traced in opposite directions. For their distance apart, measured from the last to the first, a fresh evaluation of the integral in Equation [31d] gives

$$\Delta z = i\,K \int_0^{\pi} d\theta = i\,\pi\,K$$

Here the factor i causes Δz to be perpendicular to both segments, whose directions are those of $\mp K$. This case is illustrated in Figure 38a.

Examples in which $-1 \leq \alpha_1 + \alpha_2 \ldots \alpha_n < 1$ are illustrated in Figures 38b, c, d. Here Δz, estimated as in Equation [31d], is infinite. In Figure 38d the geometrical polygon is reduced to a single semi-infinite line and its "interior" includes all the remainder of the z plane.

2. As an alternative, one of the α's may itself exceed 1. Then the integrals up to the corresponding point a, as in [31b], diverge, and both adjacent sides extend to infinity. No α should be made greater than 2, however. Two cases are shown in Figures 38e and 38f; in 38f the polygon consists of two unconnected infinite lines.

As with finite polygons, a given infinite polygon can be transformed into the real t axis in different ways.

In any case, as t traverses its real axis positively, the upper half of the t-plane lies to the left; hence, as explained under conformal mapping, the corresponding region on the z-plane lies to the left as z traces the perimeter of the polygon. In the case of finite polygons, the region on the left is the interior; with an infinite polygon, the region on the left is that

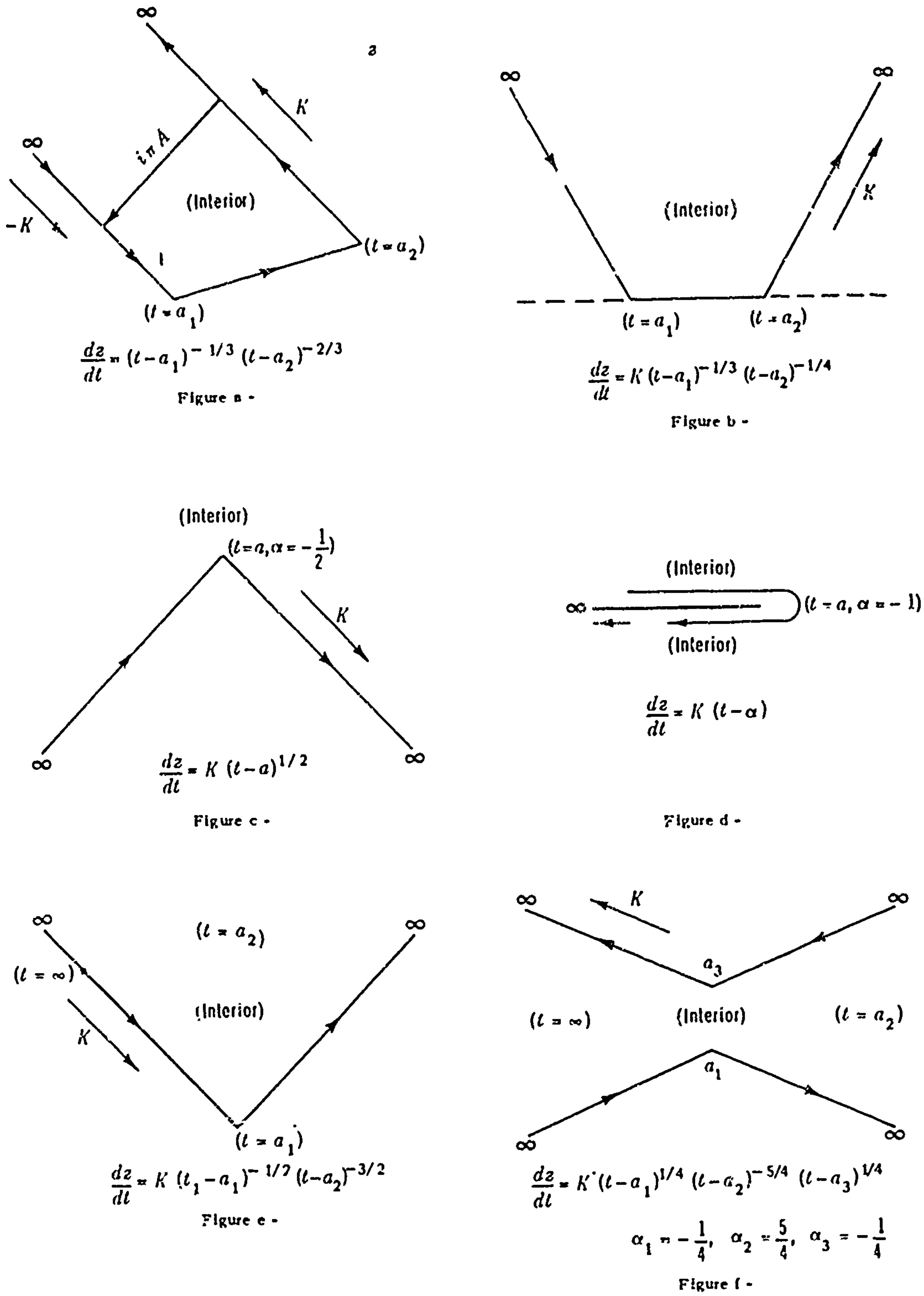


Figure 38 – Some infinite polygons.

which is transformed into the upper half of the t-plane and hence may appropriately be defined to constitute the interior. An infinite polygon can be traced in either direction; if the direction is reversed, the exterior angles are replaced by their supplements, and the former exterior be-becomes the interior.

Detailed discussions of a number of cases will be found in the next chapter.

32. THE HYPERBOLIC FUNCTIONS

The following formulas are collected here for convenience of reference. Where $\pm$ occurs twice in the same formula, the upper sign is to be taken throughout the formula, or the lower sign throughout. The positive square root is always meant, and ln denotes the logarithm to base e.

$$\sinh x = \frac{1}{2}(e^{x}-e^{-x}), \qquad \cosh x = \frac{1}{2}(e^{x}+e^{-x})$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}, \qquad \coth x = \frac{\cosh x}{\sinh x} = \frac{e^{x}+e^{-x}}{e^{x}-e^{-x}}$$

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^{x}+e^{-x}}, \qquad \operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^{x}-e^{-x}}$$

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$

$$\sinh 2x = 2\sinh x \cosh x, \quad \cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\tanh 2x = \frac{2\tanh x}{1+\tanh^2 x}, \qquad \coth 2x = \frac{1}{2}(\tanh x + \coth x)$$

$$\sinh \frac{1}{2}x = \pm\sqrt{\frac{1}{2}(\cosh x - 1)} \qquad \cosh \frac{1}{2}x = \sqrt{\frac{1}{2}(\cosh x + 1)}$$

(The sign is + or − according as the value of x is + or −)

$$\tanh \frac{1}{2}x = \frac{\sinh x}{\cosh x + 1}, \qquad \coth \frac{1}{2}x = \frac{\sinh x}{\cosh x - 1}$$

$$\sinh^{-1} x = \ln(x + \sqrt{x^2+1}), \qquad \tanh^{-1} x = \frac{1}{2}\ln\frac{1+x}{1-x}$$

$$\cosh^{-1} x = \pm\ln(x + \sqrt{x^2-1}), \qquad \coth^{-1} x = \frac{1}{2}\ln\frac{x+1}{x-1}$$

$$\frac{d}{dx}\sinh x = \cosh x \qquad \frac{d}{dx}\cosh x = \sinh x$$

$$\frac{d}{dx}\tanh x = \operatorname{sech}^2 x \qquad \frac{d}{dx}\coth x = -\operatorname{csch}^2 x$$

$$\frac{d}{dx}\operatorname{sech} x = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}\operatorname{csch} x = -\operatorname{csch} x \coth x$$

$$\sin(ix) = i \sinh x \qquad \sinh(ix) = i \sin x$$

$$\cos(ix) = \cosh x \qquad \cosh(ix) = \cos x$$

$$\tan(ix) = i \tanh x \qquad \tanh(ix) = i \tan x$$

$$\cot(ix) = -i \coth x \qquad \coth(ix) = -i \cot x$$

$$\sin(x \pm iy) = \sin x \cosh y \pm i \cos x \sinh y$$

$$\cos(x \pm iy) = \cos x \cosh y \mp i \sin x \sinh y$$

$$\tan(x \pm iy) = \frac{1}{2}\,\frac{\sin 2x \pm i \sinh 2y}{\cos^2 x + \sinh^2 y}$$

$$\cot(x \pm iy) = \frac{1}{2}\,\frac{\sin 2x \mp i \sinh 2y}{\sin^2 x + \sinh^2 y}$$

$$\sec(x \pm iy) = \frac{\cos x \cosh y \pm i \sin x \sinh y}{\cos^2 x + \sinh^2 y}$$

$$\csc(x \pm iy) = \frac{\sin x \cosh y \mp i \cos x \sinh y}{\sin^2 x + \sinh^2 y}$$

$$\sinh(x \pm iy) = \sinh x \cos y \pm i \cosh x \sin y$$

$$\cosh(x \pm iy) = \cosh x \cos y \pm i \sinh x \sin y$$

$$\tanh(x \pm iy) = \frac{1}{2}\,\frac{\sinh 2x \pm i \sin 2y}{\sinh^2 x + \cos^2 y}$$

$$\coth(x \pm iy) = \frac{1}{2}\,\frac{\sinh 2x \mp i \sin 2y}{\sinh^2 x + \sin^2 y}$$

$$\operatorname{sech}(x \pm iy) = \frac{\cosh x \cos y \mp i \sinh x \sin y}{\sinh^2 x + \cos^2 y}$$

$$\operatorname{csch}(x \pm iy) = \frac{\sinh x \cos y \mp i \cosh x \sin y}{\sinh^2 x + \sin^2 y}$$

$$\sinh^2 x \sin^2 y + \cosh^2 x \cos^2 y = \sinh^2 x + \cos^2 y$$

$$= \cosh^2 x - \sin^2 y = \frac{1}{2}(\cosh 2x + \cos 2y) \qquad [32a]$$

$$\sinh^2 x \cos^2 y + \cosh^2 x \sin^2 y = \sinh^2 x + \sin^2 y$$

$$= \cosh^2 x - \cos^2 y = \frac{1}{2}(\cosh 2x - \cos 2y) \qquad [32b]$$

The first six of these formulas may serve as definitions; the others can be deduced from them, or, for functions containing i, from results obtained in Section 20. In some cases denominators are rationalized. In the last two formulas, the second and third members may be added and divided by 2 in order to obtain the fourth.

The first four functions are plotted in Figure 39.

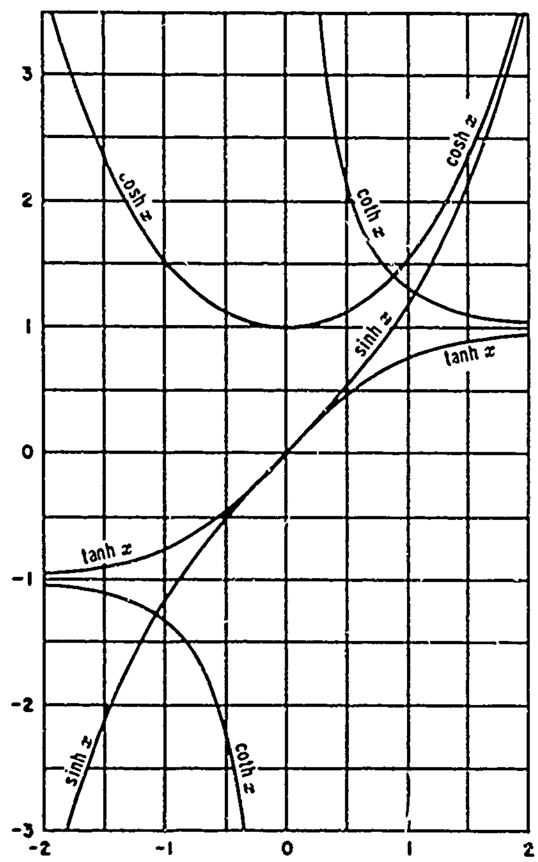


Figure 39 – Plots of four hyperbolic functions.

33. SOME SERIES

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \qquad [33a]$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \ldots \qquad [33b]$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \ldots \qquad [33c]$$

$$\tan x = x + \frac{1}{3} x^3 + \frac{2}{15} x^5 + \frac{17}{315} x^7 \ldots \quad |x| < \frac{\pi}{2} \qquad [33d]$$

$$\cot x = \frac{1}{x} - \frac{x}{3} - \frac{1}{45} x^3 - \frac{2}{945} x^5 \ldots [|x| < \pi] \qquad [33e]$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} \ldots \qquad [33f]$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} \ldots \qquad [33g]$$

$$\tanh x = x - \frac{1}{3} x^3 + \frac{2}{15} x^5 - \frac{17}{315} x^7 \ldots \quad |x| < \frac{\pi}{2} \qquad [33h]$$

$$\coth x = \frac{1}{x} + \frac{x}{3} - \frac{1}{45} x^3 \div \frac{2}{945} x^5 \ldots [|x| < \pi] \qquad [33i]$$

$$\sin^{-1} x = x + \frac{1}{2\cdot 3} x^3 + \frac{1\cdot 3}{2\cdot 4\cdot 5} x^5 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7} x^7 \ldots [|x| < 1] \qquad [33j]$$

$$\tan^{-1} x = x - \frac{1}{3} x^3 + \frac{1}{5} x^5 - \frac{1}{7} x^7 \ldots [|x| < 1]$$

$$= \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} \ldots [|x| > 1] \qquad [33k]$$

$$\sinh^{-1} x = x - \frac{1}{2\cdot 3} x^3 + \frac{1\cdot 3}{2\cdot 4\cdot 5} x^5 \ldots [|x| < 1] \qquad [33l]$$

$$\tanh^{-1} x = x + \frac{1}{3} x^3 + \frac{1}{5} x^5 + \frac{1}{7} x^7 + \ldots x^7 + \ldots [|x| < 1] \qquad [33m]$$

$$\ln (1 + x) = x - \frac{1}{2} x^2 + \frac{1}{3} x^3 - \frac{1}{4} x^4 \ldots [|x| < 1] \qquad [33n]$$

Except as indicated, the expansions hold for all values of x. See Smithsonian Mathematical Formulas[18] for other series.

CHAPTER III

CASES OF TWO-DIMENSIONAL FLOW

The principal known types of two-dimensional flow, including all that are treated in Lamb's Hydrodynamics,[1] will be described or listed in this chapter. The important formulas will be deduced and plots of the streamlines or sometimes the flow net will be shown.

As the theory of complex variables is particularly suited for two-dimensional problems, it will be used consistently. Acquaintance with the theory will be assumed, to the extent of the summary in Chapter II, and also with hyperbolic functions, for which some formulas are listed in Section 32. As a rule the standard formulation described in the next section will be adopted.

34. NOTATION AND FORM OF PRESENTATION

The given boundaries pertaining to a particular problem are assumed to be drawn on the plane of the complex variable $z = x + iy$, on which x and y are real Cartesian coordinates. Diagrams on this plane will be labeled indiscriminately with symbols representing geometrical magnitudes, such as points or distances, and with symbols representing complex numbers.

The appropriate mathematical transformation is represented in each case by

$$w(z) = \phi(x,y) + i\,\psi(x,y)$$

where ϕ and ψ are real functions of x and y. Except as stated, the fluid is assumed to be at rest at infinity. For simplicity, each transformation is regarded as giving rise to two conjugate types of flow.

In one type of flow, ϕ represents the velocity potential and ψ the stream function; the equipotential curves are given by ϕ = constant, and the streamlines by ψ = constant. The x and y components of the velocity are then

$$u = -\frac{\partial\phi}{\partial x} = -\frac{\partial\psi}{\partial y}, \quad v = -\frac{\partial\phi}{\partial y} = \frac{\partial\psi}{\partial x} \qquad [34a,b]$$

In the conjugate type of flow, the velocity potential ϕ' and the stream function ψ' are related to ϕ and ψ as follows:

$$\phi' = \psi, \quad \psi' = -\phi$$

The equipotential curves and the streamlines are interchanged; and the velocity components u', v' are:

$$u' = -\frac{\partial\phi'}{\partial x} = -v, \quad v' = -\frac{\partial\phi'}{\partial y} = u \qquad [34c,d]$$

Thus the vector velocity (u', v') is rotated positively or counterclockwise through 90 degree relative to (u, v). The second type of flow also constitutes the first type as furnished by the modified transformation,

$$w_1(z) = \phi' + i\psi' = -i\,w(z)$$

The flow net, or the pattern of equipotential curves and streamlines, is geometrically the same for both types of flow. The magnitude of the velocity is also the same in the two types, namely,

$$q = (u^2 + v^2)^{1/2} = (u'^2 + v'^2)^{1/2} = \left|\frac{dw}{dz}\right| \qquad [34e]$$

Furthermore, since $dw/dz = \partial w/\partial x = \partial\phi/\partial x + i\,\partial\psi/\partial x$,

$$-u + iv = \frac{dw}{dz}, \qquad -u' + iv' = -i\frac{dw}{dz} = \frac{dw_1}{dz} \qquad [34f,g]$$

Usually u and v are most easily found from Equation [34f] by separating dw/dz into its real and imaginary parts; in order to do this, it may be necessary to rationalize a denominator by multiplying by its complex conjugate. Frequently, values of u and v obtained in this manner will be given without writing down dw/dz. In some cases, however, use of Equations [34a, b] is more convenient.

Stagnation points occur in both types where $dw/dz = 0$ and hence $q = 0$. At such points the transformation may fail to be conformal, and equipotential curves and streamlines may meet at other angles than 90 degree.

Singular points for the transformation occur wherever $dw/dz \to \infty$. Since at such points $q \to \infty$, they must be excluded from the body of the fluid by inserting suitable boundaries. It is convenient, however, to allow a singularity to fall *on* a boundary; in a physical case, it can then always be imagined to be removed from the region of the fluid by slightly altering the shape of the boundary.

When polar coordinates r, θ are employed, the component of the velocity in the radial direction is denoted by q_r, that in the transverse direction of increasing θ by q_θ; these components are calculated as

$$q_r = -\frac{\partial\phi}{\partial r}, \qquad q_\theta = -\left(\frac{1}{r}\right)\frac{\partial\phi}{\partial\theta}$$

Many-valued functions are to be understood as defined so that they vary continuously with z, or with x and y, in all variations that are possible without crossing any boundaries that may be present. If it is appropriate in a given case to choose a single set of values for such a function, this is to be done in such manner that the function takes on its ordinary values at points on the positive x-axis, or the positive real axis for z.

The symbol $\sqrt{\ }$ will be used only for the positive square root of a positive real number.

In many figures the curves which are streamlines when ϕ is the potential will be marked with arrows. For the conjugate flow the arrows are then to be supposed transferred to the other set of curves. Curves are always drawn for equally spaced values of ϕ or of ψ, and in flow nets the same spacing is used for both ϕ and ψ. Sometimes, however, an intermediate curve, for a value midway between those for the adjacent curves, may be shown as a broken line.

Physical cases can be constructed as desired by inserting a rigid boundary along any streamline; this does not disturb the flow, since friction is assumed to be absent. If the boundary extends to infinity so as to divide the field completely, the flow can be assumed to occur only on one side of it, or to differ by a constant factor on the two sides. Special cases corresponding to different possible positions of such a boundary are not usually illustrated.

The positive direction for angles, and for tracing closed curves, is taken as usual to be counterclockwise. Thus, in tracing a closed curve positively, its interior lies on the left. This direction is understood in the symbol $\oint$, denoting the line integral around a closed curve, and in the fundamental definition of the circulation.

The circulation Γ around any closed curve is also equal to the negative of the algebraic change in the velocity potential on going once around the curve in the positive direction.

Many types of two-dimensional flow possess one or more planes of symmetry, which are represented on the xy-plane by a line of symmetry. Two types of symmetry may be distinguished.

In one type, which will be called symmetry of flow, the actual motion on one side of the plane is the mirror image of that on the other side. At points symmetrically located relative to the plane of symmetry, the values of q and ϕ are the same, also those of the pressure p, and of the component of velocity parallel to the plane; whereas the component of velocity perpendicular to the plane is oppositely directed. The difference between the value of ψ and its value on the plane, which is necessarily composed of streamlines, is equal and opposite at the two points.

In the other type of symmetry, the flow net is again geometrically symmetrical, but the motions have a different relation; p, ψ, and the vector component of velocity perpendicular to the plane of symmetry have equal values at corresponding points, whereas the component of velocity parallel to the plane, and also the algebraic excess of ϕ above its value on the plane, have equal and opposite values.

Many examples of the two types of symmetry may be found in succeeding sections. The contrast is specifically mentioned, for example, in Sections 41 and 55.

The kinetic energy of the mass of fluid that is contained between two planes parallel to the flow and unit distance apart will be denoted by T_1. Its dimensions are those of kinetic energy divided by distance or ml/t^2.

Formulas for the pressure p will not usually be given. When the boundaries are stationary and the motion of the fluid is steady, the pressure is given by the Bernoulli equation,

$$p = \frac{1}{2} \rho (U^2 - q^2) + p_\infty \qquad [34h]$$

in which ρ is the density of the fluid, assumed uniform, U is the particle velocity, and p_∞ the pressure in the fluid at infinity. In many figures the difference, $p-p_\infty$ or $\rho(U^2-q^2)/2$, along selected lines or curves is shown on an arbitrary scale. The pressure along the y-axis is sometimes plotted horizontally, with positive values toward the right. Any case in which the boundaries are in uniform translatory motion may be reduced to the corresponding case in which they are at rest by a suitable change of the frame of reference, or by imparting to everything an equal velocity in the opposite direction. Such a change does not alter the distribution of pressure or the forces.

Results will commonly be stated in terms of a particular choice of axes, and sometimes in terms of particular units of length and of velocity. The use of special units permits the mathematical developments to be made in compact form; but the equations may not be dimensionally balanced. It is to be understood that the formulas, if too specialized, may always be generalized by substituting $k_1 z, k_1 x, k_1 y, k_1 r$ for z, x, y, r, also $k_2 w, k_2 \phi, k_2 \psi$, for w, ϕ, ψ, and $k_2 u/k_1$, $k_2 v/k_1$, $k_2 q/k_1$, for u, v, q, where k_1 and k_2 are any real numbers, provided these changes are made consistently in all formulas. Velocities are thereby changed in the ratio k_1/k_2, since u, v, q are then given by the original expressions each multiplied by k_1/k_2; and all linear dimensions similarly become $1/k_1$ times as great. Even the velocity at infinity is changed in the ratio k_1/k_2; and the kinetic energy in a layer of unit thickness perpendicular to the planes of flow or T_1 is multiplied in proportion to velocity squared times area or by $1/k_2^2$. The dimensional balance may then be restored, if desired, by assigning the proper dimensions to k_1 and k_2. One type of change without the other may be made by letting either k_1 or k_2 be unity.

In addition, of course, the axes may be moved into any other position by means of the usual formulas. The method of doing this in terms of z is important and was explained in Section 25. To displace the flow and all boundaries through distances h_1 in the x-direction and h_2 in the y-direction, without rotation, it suffices to replace x by $x-h_1$ and y by $y-h_2$ in all formulas, or z by $z-h$ where $h = h_1 + ih_2$. To rotate everything through an angle α about the origin, which requires rotation of the axes in the opposite direction relatively to the flow field, replace x by $x \cos\alpha + y \sin\alpha$ and y by $-x \sin\alpha + y \cos\alpha$ or z by $ze^{-i\alpha}$, and u and v, therefore, by $u \cos\alpha + v \sin\alpha$ and $-u \sin\alpha + v \cos\alpha$, respectively, in all formulas. To effect first the rotation, about the initial origin, then the displacement, substitute $(z-h)e^{-i\alpha}$ for z; or, if $z = f(w)$, take $z = h + e^{i\alpha} f(w)$.

Where $w(z)$ contains a real multiplicative constant, often A or U, it is to be understood that reversal of the sign of this factor merely reverses all velocities, with an accompanying change of sign of ϕ and ψ but without any change in the geometrical equipotential curves and streamlines and without change of the pressure. Arrows drawn on the streamlines in the plots refer in each case to a positive value of this constant.

It should be remembered that states of flow of an incompressible fluid may be superposed freely to form new states of flow. The potential, stream functions, and velocity components add algebraically, the velocities themselves, vectorially. The pressures and forces, however, are not additive. Out of all the cases that can be constructed in this manner, only the most interesting will be mentioned.

The section on units in Chapter I, Section 18, may be noted.

The older convention as to the signs of ϕ and ψ described in Sections 6, 13, and 16, may also be noted; it is often encountered in the literature. Formulas based on this older convention may be obtained by changing the sign before every symbol representing a velocity component.

SOME SIMPLE TYPES OF FLOW

35. UNIFORM MOTION

$$w = Az + C, \qquad A = a + ib$$

where a and b are real constants and C is another constant, real or complex. This transformation was discussed mathematically in Section 24. Since $w = \phi + i\,\psi$,

$$\phi = ax - by, \qquad \psi = bx + ay$$

$$u = -a, \quad v = b, \quad q = \left|\frac{dw}{dz}\right| = |A| = (a^2 + b^2)^{1/2}$$

The flow is thus one of uniform translation. The flow net, illustrated in Figure 40, consists of straight lines.

If the fluid is moving at velocity U in a direction inclined at an angle α to the negative x-axis, as in Figure 41, $A = U$, $u = -a = -U\cos\alpha$, $v = b = -U\sin\alpha$, and, with omission of the physically meaningless constant C,

$$w = U(\cos\alpha - i\sin\alpha)\,z = Uze^{-i\alpha} \qquad [35a]$$

(For notation and method; see Section 34; Reference 2, Section 6.0.)

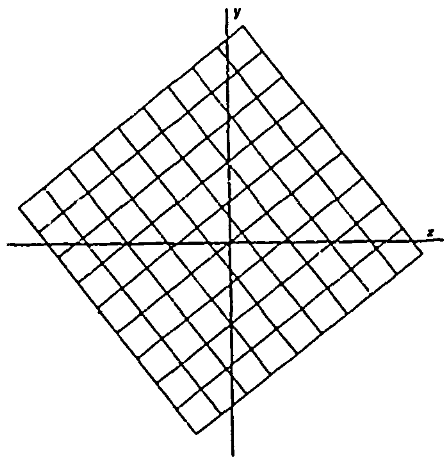


Figure 40 – Flow net for uniform flow.

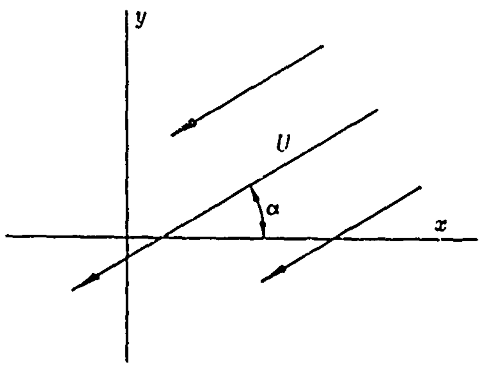


Figure 41 – Definition of U and α in Equation [35a].

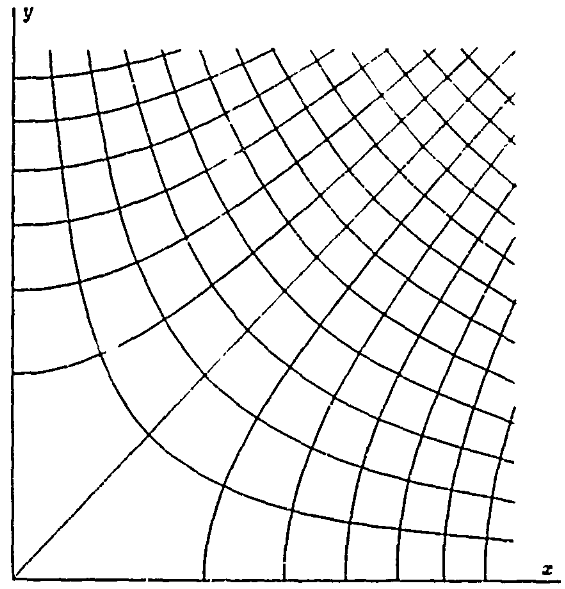


Figure 42 – Flow net in a right angle: $w = Az^2$.

36. HYPERBOLIC FLOW

$$w = A z^2 \qquad [36a]$$

where A is a real constant. Then $\phi + i\,\psi = A(x + iy)^2$,

$$\phi = A(x^2 - y^2), \quad \psi = 2Axy \qquad [36b,c]$$

Both the equipotential curves and the streamlines are rectangular hyperbolas, the former with asymptotes inclined at 45 degree to the x- y-axes, the latter with the axes themselves as asymptotes.

In the conjugate flow

$$\phi' = 2Axy, \qquad \psi' = A(y^2 - x^2) \qquad [36d,e]$$

In both cases $q = \infty$ at infinity, $q = 0$ at the origin.

The flow net is the same in all quadrants. One quadrant is shown in Figure 42. It may be used to give an approximate idea of the flow past a square corner.

The two conjugate flows differ only in that the flow pattern is rotated through 45 degree. (For notation and method; see Section 34; Reference 1, Article 63; Reference 2, Section 4.70.)

37. LINE DIPOLE

$$w = \frac{\mu}{z}\text{ , }\mu\text{ is a real constant,} \qquad [37a]$$

$$\phi + i\psi = \frac{\mu}{x+iy} = \frac{\mu(x-iy)}{(x+iy)(x-iy)} = \frac{\mu x}{x^2+y^2} - i\,\frac{\mu y}{x^2+y^2}\,,$$

$$\phi = \frac{\mu x}{x^2+y^2}, \qquad \psi = -\frac{\mu y}{x^2+y^2} \qquad [37b,c]$$

$$q = \left|\frac{dw}{dz}\right| = \mu \left|-\frac{1}{z^2}\right| = \mu \left|\frac{-1}{x^2 - y^2 + 2\,ixy}\right| = \frac{\mu}{\left[(x^2-y^2)^2 + 4\,x^2\,y^2\right]^{1/2}} \qquad [37d]$$

or

$$q = \frac{\mu}{x^2+y^2} = \frac{\mu}{r^2}, \; r = (x^2+y^2)^{1/2} \qquad [37e,f]$$

Thus $q = \infty$ at the origin, where $z = 0$ and a singularity occurs.

In terms of polar coordinates r and θ, with θ measured from the x-axis so that $x = r\cos\theta$, $y = r\sin\theta$,

$$\phi = \frac{\mu\cos\theta}{r}, \qquad \psi = -\frac{\mu\sin\theta}{r} \qquad [37g,h]$$

The equipotential curves and the streamlines are circles through the origin; their equations are obtained by assigning a constant value to ϕ or ψ in Equations [37b,c] or in the equivalent equations

$$\left(x - \frac{\mu}{2\phi}\right)^2 + y^2 = \frac{\mu^2}{4\phi^2}, \; x^2 + \left(y + \frac{\mu}{2\psi}\right)^2 = \frac{\mu^2}{4\psi^2} \qquad [37i,j]$$

The radius is $\mu/(2\,|\phi|)$, or $\mu/(2|\psi|)$. See Figure 43.

This is the flow due to a uniform line dipole or doublet. It is obtained in more elementary fashion in Section 15. The axis of the dipole is here the x-axis, which represents a plane of symmetry. The constant μ represents twice the point-dipole moment per unit length; it may be called the line-dipole moment. The dipole axis is regarded as directed toward the side of maximum ϕ: if $\mu > 0$, this is here the positive x-axis, if $\mu < 0$, it is the negative x-axis.

The components of velocity in the directions of x and y, or of r and θ, respectively, are

$$u = \mu\frac{x^2-y^2}{r^4}, \quad v = \frac{2\mu xy}{r^4}; \quad q_r = \mu\frac{\cos\theta}{r^2}, \; q_\theta = \mu\frac{\sin\theta}{r^2} \qquad [37k,l,m,n]$$

The *conjugate* flow represents a line dipole with axis along the y-axis, directed toward negative y if $\mu > 0$. It is also obtainable from the transformation

$$w = -\frac{i\mu}{z}, \; \phi = -\frac{\mu y}{r^2}, \quad \psi = -\frac{\mu x}{r^2} \qquad [37o,p,q]$$

More generally, the transformation

$$w = \frac{\mu e^{i\alpha}}{z - z_0}, \qquad [37r]$$

where μ and α are real and $z_0 = x_0 + iy_0$, represents a line dipole located on a line cutting the xy-plane at (x_0, y_0), with its axis inclined at an angle α to the positive x-axis; see Figure 44. Since

$$\frac{e^{i\alpha}}{z - z_0} = \frac{\cos\alpha + i\sin\alpha}{x - x_0 + i\,(y - y_0)} = \frac{(\cos\alpha + i\sin\alpha)\,[x - x_0 - i(y - y_0)]}{(x-x_0)^2 + (y-y_0)^2}$$

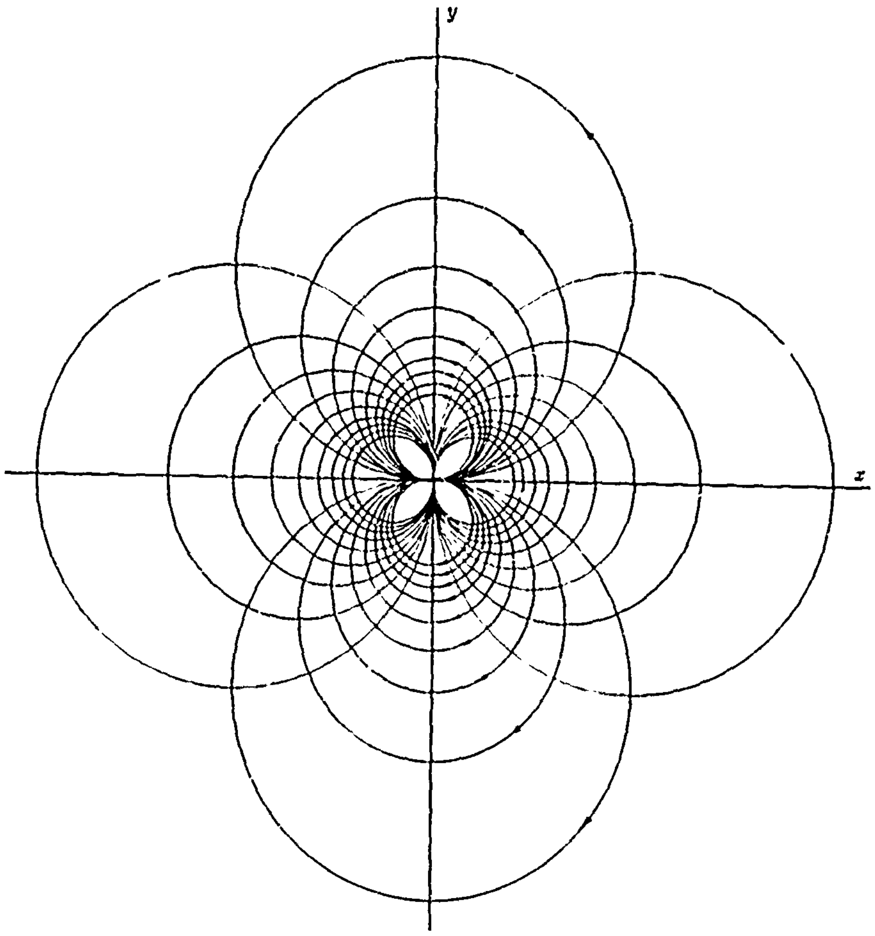


Figure 43 – Flow net for a line dipole: $w = \mu/z$.

by substituting in terms of polar coordinates with origin at (x_0, y_0) so that

$$r = [(x-x_0)^2 + (y-y_0)^2]^{1/2},\ x-x_0 = r\cos\theta,\ y-y_0 = r\sin\theta$$

it is found that

$$\phi = \mu\frac{\cos(\theta-\alpha)}{r},\quad \psi = -\mu\frac{\sin(\theta-\alpha)}{r} \qquad [37s,t]$$

$$q_r = \mu\frac{\cos(\theta-\alpha)}{r^2},\quad q_\theta = \mu\frac{\sin(\theta-\alpha)}{r^2} \qquad [37u,v]$$

The singularity now occurs at $z = z_0$, or at (x_0, y_0), and the plane of symmetry passes through x_0, y_0 and is inclined at an angle α to the x-axis; these facts verify the statement made as to the location and orientation of the dipole.

The geometrical properties of the transformation are discussed in Section 24.

(For notation and method; see Section 34; Reference 1, Article 63; Reference 2, Section 8.23.)

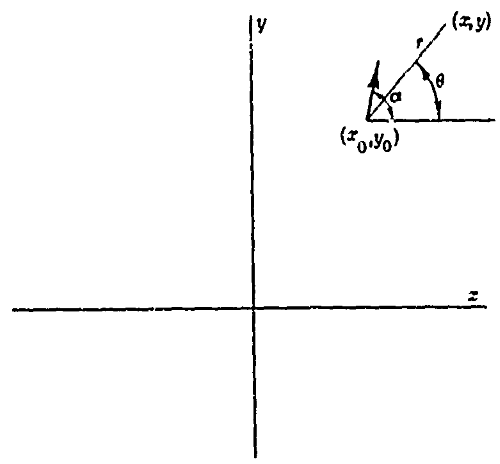


Figure 44 - Definition of α, x_0, y_0, in Equation [37r].

38. LINE QUADRUPOLE

$$w = \frac{A}{z^2}$$

where A is a real constant. If $z = re^{i\theta}$,

$$\phi + i\psi = A\left(re^{i\theta}\right)^{-2} = \frac{A}{r^2}\left(\cos 2\theta - i \sin 2\theta\right) \qquad [38a]$$

$$\phi = \frac{A}{r^2}\cos 2\theta,\ \psi = -\frac{A}{r^2}\sin 2\theta,\ \theta = \tan^{-1}\frac{y}{x} \qquad [38b,c]$$

$$-u + iv = \frac{dw}{dz} = -\frac{2A}{r^3}e^{-3i\theta} = -\frac{2A}{r^3}(\cos 3\theta - i \sin 3\theta) \qquad [38d]$$

hence

$$u = \frac{2A}{r^3}\cos 3\theta,\ v = \frac{2A}{r^3}\sin 3\theta \qquad [38e,f]$$

$$q = \left|\frac{dw}{dz}\right| = \frac{2A}{r^3};\ r = (x^2+y^2)^{1/2} \qquad [38g,h]$$

This type of flow can be produced in the limit out of that due to two opposing dipoles placed close together, hence the name quadrupole. The origin is a singular point, a pole of the second order.

The equipotential curves and streamlines are lemniscates. The first quadrant of the flow net is illustrated in Figure 45; other quadrants are geometrically similar, with changes of sign in ϕ and ψ that are easily determined.

(For notation and method; see Section 34; Reference 1, Article 63.)

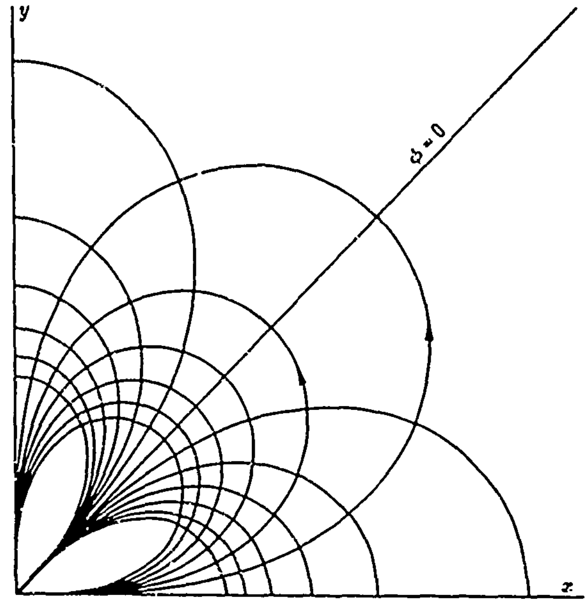


Figure 45 – Part of flow net for a line quadrupole: $w = A/z^2$.

39. FLOW IN AN ANGLE

$$w = Az^{\pi/\alpha} \qquad [39a]$$

where A and α are real constants and α is positive. If $z = re^{i\theta}$,

$$\phi + i\psi = A(re^{i\theta})^{\pi/\alpha} = Ar^{\pi/\alpha}\left(\cos \pi \frac{\theta}{\alpha} + i \sin \pi \frac{\theta}{\alpha}\right)$$

$$\phi = Ar^{\pi/\alpha} \cos \pi \frac{\theta}{\alpha}, \quad \psi = Ar^{\pi/\alpha} \sin \pi \frac{\theta}{\alpha} \qquad [39b,c]$$

$$-u + iv = \frac{dw}{dz} = A\frac{\pi}{\alpha} z^{\frac{\pi}{\alpha} - 1}, \qquad q = A\frac{\pi}{\alpha} r^{\frac{\pi}{\alpha} - 1} \qquad [39d,e]$$

The origin is a singular point, unless π/α is an integer. If $\alpha > \pi$, dw/dz becomes infinite at $z = 0$. In any case, as z goes round the point $z = 0$, the amplitude of w increases by $(2\pi)\pi/\alpha$ and that of dw/dz by $(2\pi)(\pi/\alpha - 1)$. Hence, if π/α is not an integer, both w and dw/dz are multiple-valued in the neighborhood of $z = 0$ or $x = y = 0$. In applications, therefore, a boundary must be introduced excluding the origin and also extending to infinity, in order to make dw/dz and the components of the velocity single-valued.

The diagram of the equipotentials is the same as that of the streamlines but rotated through an angle $\alpha/2$.

The principal application is to represent the flow between two planes meeting at an angle of α radians. On one plane let $\theta = 0$ and on the other $\theta = \alpha$; then $\psi = 0$ on both planes, and they cut the xy-plane along a streamline. On these planes $\phi = \pm Ar^{\pi/\alpha}$.

One sector of the streamline diagram for $\alpha = \pi/3$ is shown in Figure 46; that for $\alpha = 3\pi/2$ is shown in Figure 47.

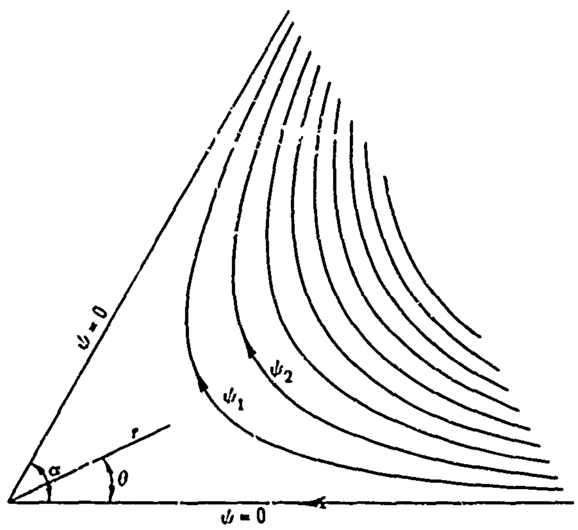


Figure 46 – Streamlines in an angle of $\alpha = \pi/3$ radians: $w = Az^3$.

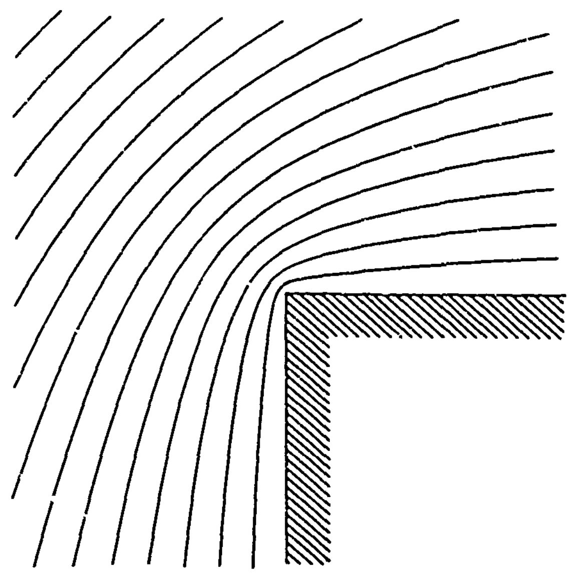


Figure 47 – Streamlines around a right angle: $\alpha = 3\pi/2$, $w = z^{2/3}$.

If $\alpha = 2\pi$, the flow is around the edge of a semi-infinite plane. In this case

$$w = Az^{1/2}, \quad \phi = Ar^{1/2} \cos\frac{\theta}{2}, \quad \psi = Ar^{1/2} \sin\frac{\theta}{2}, \quad q = \frac{A}{2} r^{-\frac{1}{2}} \qquad [39f,g,h,i]$$

Streamlines for this case are shown in Figure 48.

A uniform flow parallel to the plane may be added, producing streamlines as shown in Figure 49; see Cisotti, Reference 24.

The mathematical transformation $z' = z^n$, where n is real, is useful in constructing transformations for special purposes. Geometrically, it merely rotates all radii from the origin, except the positive real axis, about the origin as center until on the z'-plane they make an angle with the positive real axis n times as great as on the z-plane. The change is like the opening or shutting of a fan. If n is an integer, the z-plane is mapped n times onto the z'-plane; the mapping is backwards if n is negative. If $|n| < 1$, the entire z-plane is mapped onto a sector of angle $2\pi n$ radians. If n is not integral, the transformation is many-valued, with $z = 0$ as a branch point. In any case, circles centered at the origin transform into arcs of similar circles.

The more general transformation $z' = Cz^n$ also stretches all radii from the origin in a ratio equal to $|C|$ and rotates everything through an additional angle equal to amp C.

(For notation and method; see Section 34; Reference 1, Article 63; Reference 2 Section 6.0.)

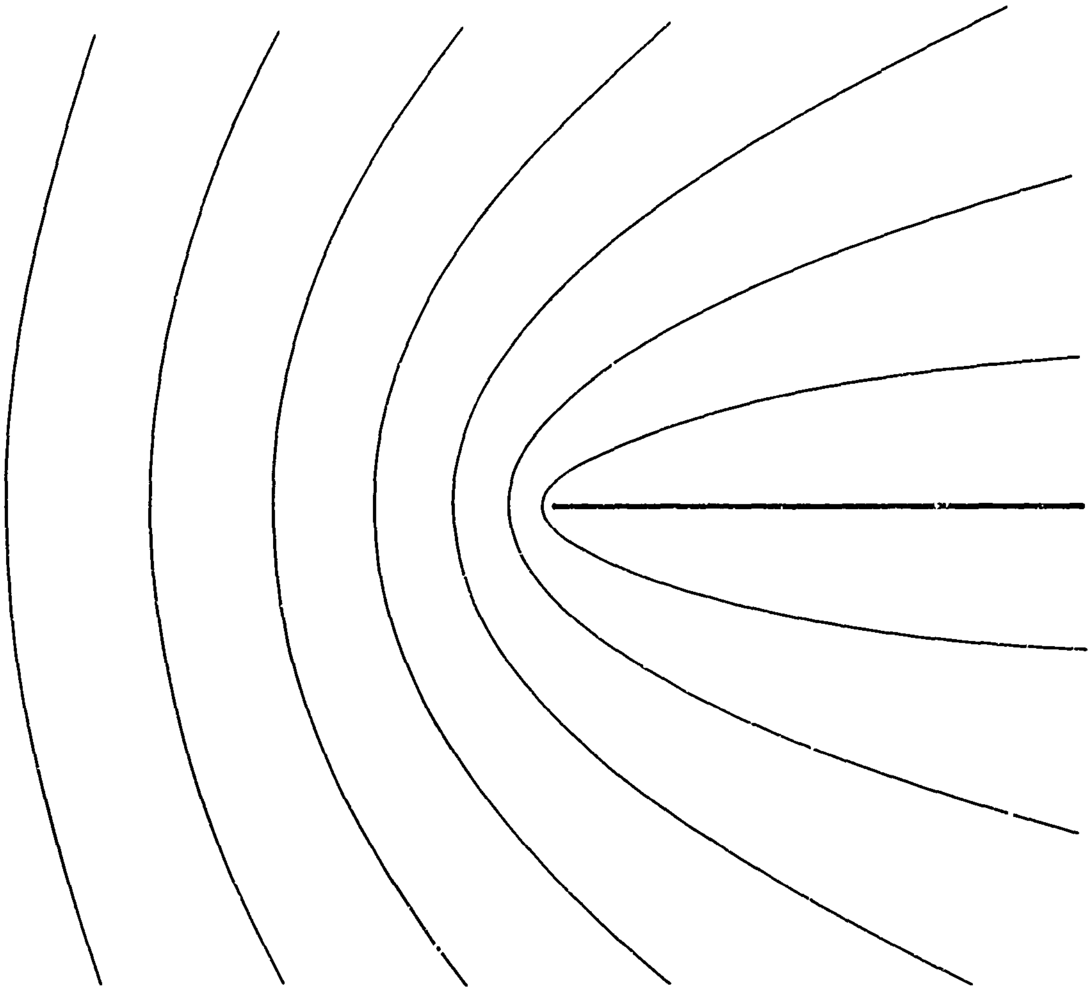


Figure 48 – Symmetrical streamlines around the edge of a semi-infinite plane: $\alpha = 2\pi$, $w = Az^{1/2}$.

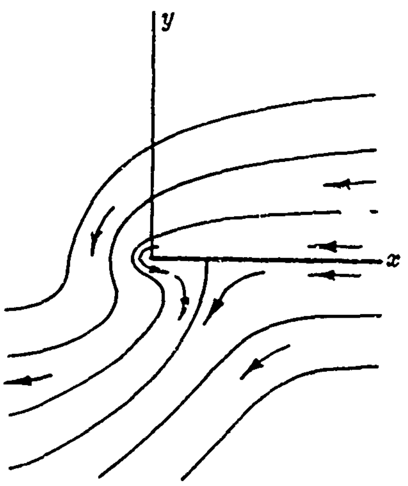


Figure 49 – Asymmetric streamlines around the edge of a semi-infinite plane. (Copied from Reference 24.)

40. LOGARITHMIC FLOW

$$w = -A \ln z \qquad [40a]$$

Let A be real. Then, if $z = x + iy = re^{i\theta}$

$$\phi + i\psi = -A \ln (re^{i\theta}) = -A \ln r - iA\theta$$

$$\phi = -A \ln r, \; \psi = -A\theta \qquad [40b,c]$$

where $r = (x^2 + y^2)^{1/2}$, $\theta = \tan^{-1} y/x$;

$$-u + iv = \frac{dw}{dz} = -\frac{A}{z} = -\frac{A}{r} e^{-i\theta} \qquad [40d]$$

$$u = \frac{A}{r} \cos\theta, \quad v = \frac{A}{r} \sin\theta, \quad q = \frac{A}{r} \qquad [40e,f,g]$$

The origin, $z = 0$ or $x = y = 0$, is a singular point.

Line Source

The equipotential curves defined by ϕ = constant are circles about the origin, each defined by a constant value of r or by

$$x^2 + y^2 = e^{-2\phi/A}$$

The streamlines, defined by ψ = constant, are radial lines from the origin; see Figure 50.

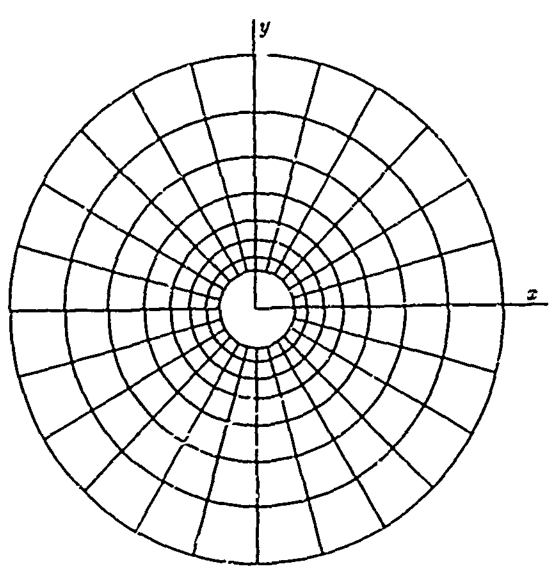


Figure 50 – Flow net for a line source or vortex: $w = -A \ln z$
(Copied from Reference 7)

This is the flow due to a uniform line source, as described in Section 15. The volume emitted by the source per second, per unit of its length, is $2\pi rq = 2\pi A$. The velocity becomes infinite as the origin is approached and is undefined at the origin itself. This type of flow is physically impossible in incompressible, indestructible fluid, but it is useful mathematically in building up by super-position the solutions for more complicated problems.

The stream function ψ is many-valued. In going around the origin in the positive direction, θ increases by 2π and ψ decreases by $2\pi A$. Thus the volume of fluid emitted by the source per second and per unit length, represented by $2\pi A$, is equal to the decrease in ψ upon going positively around any closed curve that encircles the source. The decrease in ψ is also equal to the volume of fluid that flows outward across the cylinder represented by such a curve, between two planes of flow unit distance apart.

It may be noted that, if the flow due to a line source located at a point P is superposed upon another flow in which the velocity at P is finite, the resultant streamlines approximate more and more to those due to the source alone as P is approached, since in the flow due to the source $q \to \infty$ at P.

The equations could be balanced dimensionally by writing $w = -A \ln (z/a)$ where a is a constant having the dimension of length. Then r is replaced by r/a in ϕ, which merely adds a constant to all values of ϕ.

Line Vortex

For the conjugate flow the circles become the streamlines and the radii the equipotentials. The potential ϕ', stream function ψ', and velocity are given by

$$\phi' = -A\theta, \quad \psi' = A \ln r \qquad [40h,i]$$

$$u' = -\frac{A}{r}\sin\theta, \quad v' = \frac{A}{r}\cos\theta, \quad q = \frac{A}{r} \qquad [40j,k,l]$$

The corresponding complex potential is

$$w = \phi' + i\psi' = iA \ln z \qquad [40m]$$

It is now the potential ϕ' that is many-valued; in going counterclockwise once around the origin, ϕ' changes by $-2\pi A$. The velocity, however, is single-valued, as is dw/dz, except at the origin; for $\phi + 2n\pi$ has the same space derivatives at any point as has ϕ itself. The circulation, taken around any closed curve encircling the origin once, is $\Gamma = 2\pi A$ Treated as an ideal case, the flow may be regarded as due to a line vortex at the origin, as described in Section 15.

In this type of flow, the singular point can be excluded by inserting a cylindrical boundary along any one of the circular streamlines. Then ϕ' and ψ' represent a physically possible irrotational circulatory motion about this cylinder. The circulation vanishes taken around any closed path that does not enclose the cylinder. If the path goes positively once around the cylinder, however, the circulation Γ around it is $2\pi A$. Thus the constant

$$A = \frac{\Gamma}{2\pi}$$

If the flow is steady, the pressure is given as usual by the Bernoulli equation.

The velocity increases without limit as the vortex is approached, and is undefined at the location of the vortex itself. Hence, if the flow due to a line vortex at P is superposed upon another flow in which the velocity is finite at P, the resultant streamlines near P

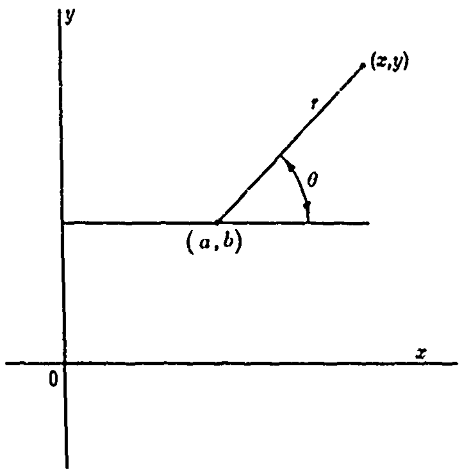


Figure 51 – Symbols for source or vortex at (a,b).

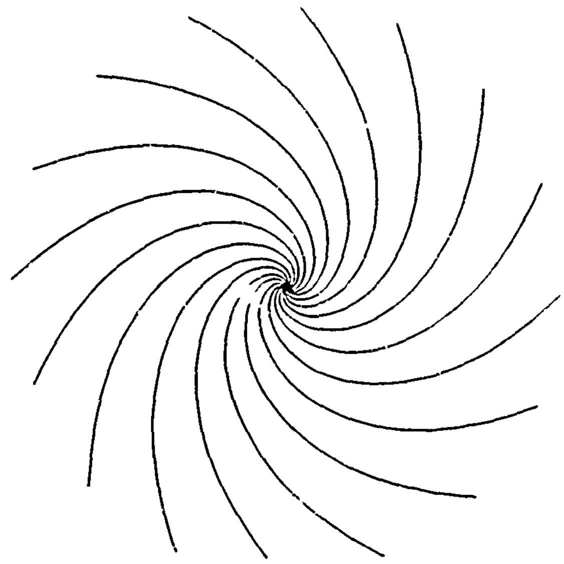


Figure 52 – Streamlines for a superposed line source and vortex: $w = -(A_1 - iA_2)\ln z$.

approximate those due to the vortex alone and consist of closed loops surrounding P; as P is approached, these loops approximate circles centered at P.

To locate the source or vortex at (a,b) instead of at the origin, it is only necessary to replace in the formulas z by $z-a-ib$, hence x by $x-a$ and y by $y-b$, and to write

$$r = [(x-a)^2 + (y-b)^2]^{½}, \quad \theta = \tan^{-1}[(y-b)/(x-a)]$$

Thus θ is measured from a line drawn in the direction of positive x, as illustrated in Figure 51.

Combined Source and Vortex

A line source and vortex may be imagined to coexist on the same line. The combined potential and stream function and the resultant velocity may be written

$$\phi = -A_1 \ln r - A_2 \theta, \quad \psi = -A_1 \theta + A_2 \ln r \qquad [40n,o]$$

$$q_r = \frac{A_1}{r}, \quad q_\theta = \frac{A_2}{r}, \quad q = \frac{1}{r}\left(A_1^2 + A_2^2\right)^{½} \qquad [40p,q,r]$$

The corresponding complex potential is $w = -(A_1 - iA_2)\ln z$. The streamlines are equiangular spirals defined by $\theta = (A_2/A_1)\ln r$ + constant, as illustrated in Figure 52. The equipotential curves are a similar set of spirals turning in the opposite direction.

Rigid walls might be inserted along any one of the spirals on which ψ is constant. If walls are inserted along n of them, chosen to be equally spaced about the axis, a first step is taken toward the idealization of a radial centrifugal pump; see Section 97.

A different type of flow between such walls, in which there is no source on the axis and hence no net outflow of fluid, was derived by Löwy from the transformation $w = z^{m+in}$ where m and n are real constants.

Geometrical Properties of the Transformation $z' = \ln z$

If $z = re^{i\theta}$ and $z' = x' + iy'$, then $x' = \ln r$, $y' = \theta$. If θ is kept in the range $-\pi < \theta \leqslant \pi$, the entire z-plane is mapped onto a horizontal strip of the z'-plane extending from but not including $y' = -\pi$ up to and including $y' = \pi$. The negative half of the real axis of z is mapped onto the upper edge of the strip at $y' = \pi$; the positive half becomes the parallel line $y' = 0$. All radii from the origin of z, in fact, become lines parallel to the real axis of z', each defined by a certain value of θ or y'. Circles about the origin of z, on the other hand, being defined by fixed values of r or of x' become lines parallel to the imaginary axis of z'; the annulus between two such circles becomes a strip in the same direction. All other straight lines on the z'-plane correspond on the z-plane to logarithmic spirals with focus at the origin.

The transformation can be visualized by imagining the z-plane to be cut just below the negative x-axis and partially shut up like a fan, while the origin is spread out over a width 2π and simultaneously displaced to minus infinity.

By selecting for θ a different range of magnitude 2π, the strip may be displaced vertically into any other position. Or, if θ is restricted to a range of width α, where $0 < \alpha < 2\pi$, the corresponding sector of apical angle α is transformed into a horizontal strip of width α on the z'-plane. Finally, if θ is allowed to range without limit, the z-plane is mapped once on every successive strip of width 2π. The complete transformation is thus infinitely many-valued.

Sometimes it is convenient in such cases to include both boundaries of the transformed area. Thus, if $-\pi \leqslant \theta \leqslant \pi$, the negative x-axis is used twice; with $\theta = -\pi$ it transforms into the lower boundary of the strip at $y' = -\pi$, with $\theta = \pi$, into the upper boundary at $y' = \pi$.

A simple closed curve not surrounding the origin on the z-plane becomes a simple closed curve on the z'-plane, but one that surrounds the origin becomes an endless curve that is periodic in the y' direction, with a period 2π, provided amp z is allowed to increase indefinitely as the z curve is traced repeatedly in the same direction. If amp z is restricted to a limited range, a closed curve about the z' origin becomes an open one on the z-plane, traced once for each traversal of the z' curve.

The more general transformation, $z' = A \ln (az) = A \ln z + A \ln a$, includes also rotation about the origin through an angle equal to amp A, a uniform change of scale in the ratio $|A|$, and the displacement represented by $A \ln a$.

(For notation and method; see Section 34; Reference 1, Article 64; Reference 2, Section 8.11, 13.10, 13.20, 13.21, 13.33.)

41. LINE SOURCE AND SINK; LINE VORTEX PAIR

$$w = A\left[\ln(z+c) - \ln(z-c)\right] \qquad [41a]$$

Or,

$$z = c \coth \frac{w}{2A}$$

For simplicity, let the constants A and c be real.

Writing

$$z - c = r_1 e^{i\theta_1}, \quad z + c = r_2 e^{i\theta_2}$$

as illustrated in Figure 53, and $z = x + iy$,

$$w = \phi + i\psi = A \ln \frac{r_2}{r_1} + i A(\theta_2 - \theta_1),$$

$$\phi = A \ln \frac{r_2}{r_1}, \quad \psi = -A(\theta_1 - \theta_2) = -A \tan^{-1} \frac{2cy}{x^2 + y^2 - c^2}, \qquad [41b,c]$$

$$-u + iv = \frac{dw}{dz} = A\left(\frac{e^{-i\theta_2}}{r_2} - \frac{e^{-i\theta_1}}{r_1}\right)$$

$$u = A\left(\frac{\cos\theta_1}{r_1} - \frac{\cos\theta_2}{r_2}\right) = 2Ac\,\frac{x^2 - y^2 - c^2}{r_1^2 r_2^2} \qquad [41d]$$

$$v = A\left(\frac{\sin\theta_1}{r_1} - \frac{\sin\theta_2}{r_2}\right) = \frac{4Ac\,xy}{r_1^2 r_2^2} \qquad [41e]$$

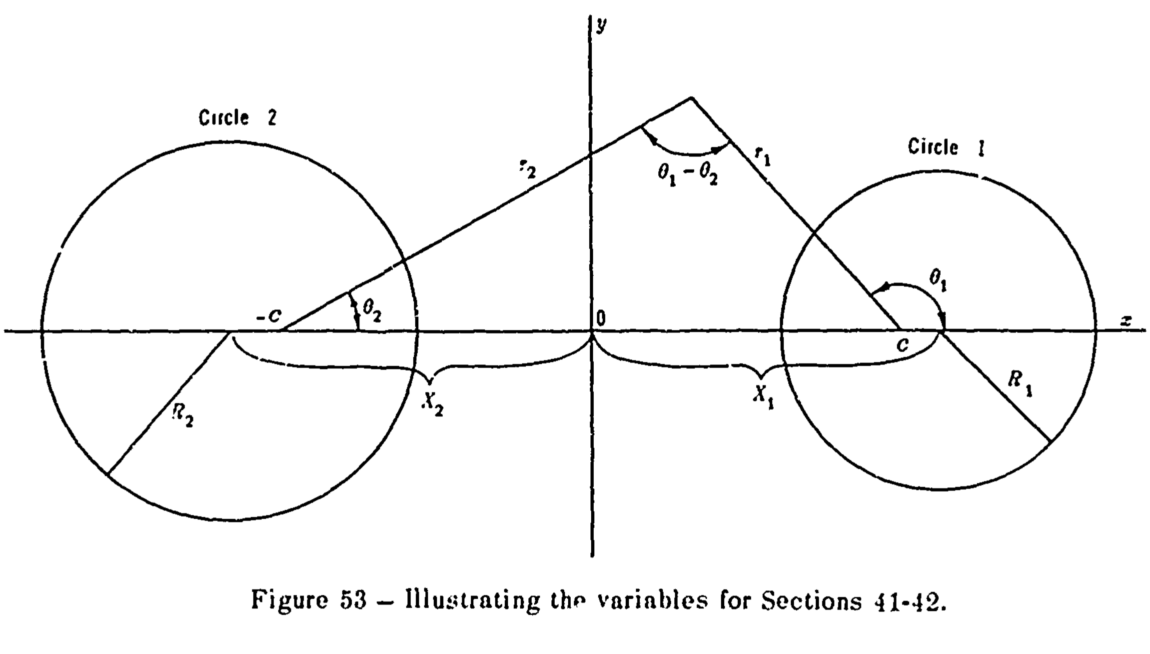


Figure 53 – Illustrating the variables for Sections 41-42.

where

$$r_1 = \left[(x-c)^2 + y^2\right]^{1/2}, \quad r_2 = \left[(x+c)^2 + y^2\right]^{1/2} \qquad [41f,g]$$

Here $\sin\theta_1 = y/r_1$, $\cos\theta_1 = (x-c)/r_1$, $\sin\theta_2 = y/r_2$, $\cos\theta_2 = (x+c)/r_2$ from which $\tan(\theta_2 - \theta_1)$ may be found. Also

$$q = (u^2 + v^2)^{1/2} = \frac{2cA}{r_1 r_2} \qquad [41h]$$

since

$$(x^2 - y^2 - c^2)^2 + 4x^2y^2 = [(x+c)^2 + y^2][(x-c)^2 + y^2]$$

On the x-axis, $u = 2Ac/(x^2 - c^2)$; on the y-axis, $u = -2Ac/(y^2 + c^2)$.

Singularities occur at $(c,0)$ and at $(-c,0)$, where $dw/dz \to \infty$.

The ϕ curves are circles with centers on the x-axis, each enclosing one of the singular points. The ψ curves are circular arcs with centers on the y-axis and ending at the singular points. The equations of these curves can be written either

$$\frac{r_2}{r_1} = e^{\phi/A}, \quad \theta_1 - \theta_2 = -\frac{\psi}{A} \qquad [41i,j]$$

or

$$\left(x - c\coth\frac{\phi}{A}\right)^2 + y^2 = c^2\,\mathrm{csch}^2\frac{\phi}{A} \qquad [41k]$$

$$x^2 + \left(y + c\cot\frac{\psi}{A}\right)^2 = c^2\csc^2\frac{\psi}{A} \qquad [41l]$$

The first of these equations in x and y is obtained by substituting from [41f,g], squaring, dividing by $e^{\phi/A}$, and rearranging. The second equation comes from the second expression given for ψ.

The flow net is illustrated in Figure 54.

The curve for $\phi = 0$ is the y-axis. If $A > 0$, $\phi \to +\infty$ at (c,0) and $-\infty$ at $(-c,0)$.

The function ψ is many-valued with a period $2\pi\,|A|$. As the point (x,y) goes positively once around $(c,0)$ without encircling $(-c,0)$, θ_1 increases by 2π and ψ changes by $\Delta\psi = -2\pi A$; if the point encircles $(-c,0)$ instead, $\Delta\theta_2 = 2\pi$ and $\Delta\psi = 2\pi A$. If, however, both singular points are encircled, or neither, then $\Delta\psi = 0$.

The curve for $\psi = 0$ (or $-2\pi A$ or $2\pi n A$) is the x-axis outside of $(c,0)$ and $(-c,0)$, provided θ_1 and θ_2 are measured from the same reference radius, as shown in Figure 53. Assume $A > 0$. Then successively smaller values of ψ are represented by the circular arcs above the axis taken in descending order. On the x-axis between (c,0) and $(-c,0)$ $\theta_1 = \pi$, $\theta_2 = 0$, $\psi = -\pi A$. The arcs below the x-axis represent successively smaller values of ψ down to $-2\pi A$, for which the curve is again the outlying part of the x-axis.

When ϕ is the potential, so that Equations [41d,e] hold, a uniform line source occurs at each singular point, one being a positive source and the other an equal sink; if $A > 0$, the

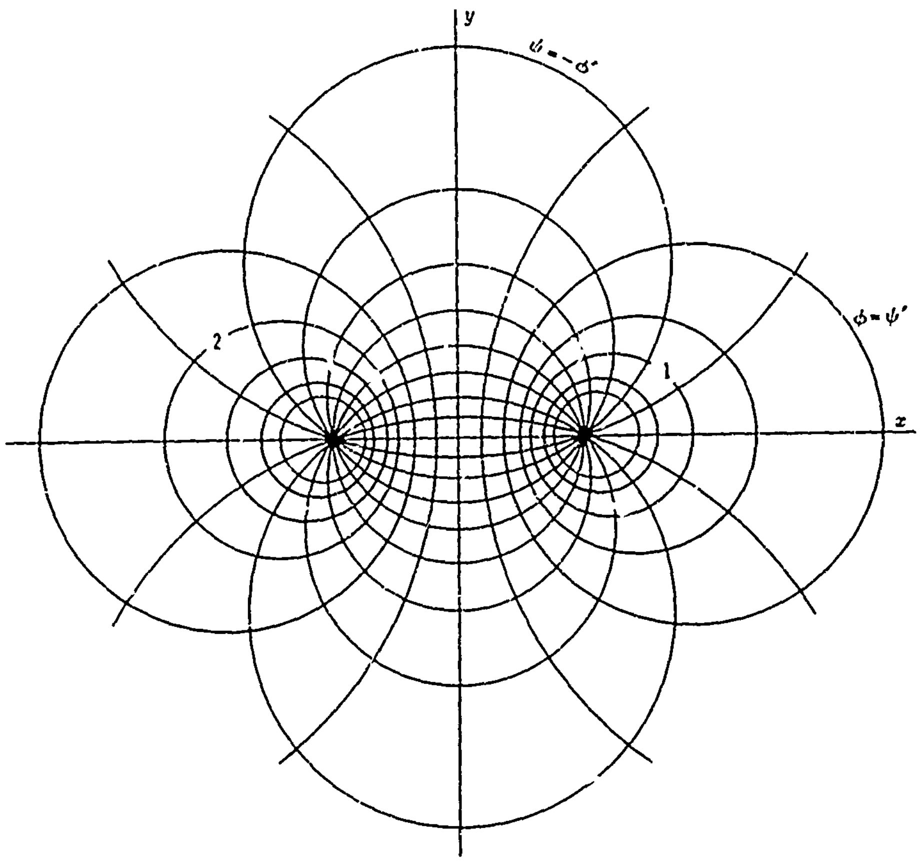


Figure 54 – Flow net for equal line source and sink, or for a pair of equal and opposite line vortices, or, in Section 42(A), for the circulating flow around and between two circular cylinders.
(Copied from Reference 1.)

positive source is at $(c,0)$. The streamlines run from the source to the sink, and the stream function ψ is many-valued. There is flow symmetry about the x-axis, mere geometrical symmetry about the y-axis.

Vortex Pair

In the *conjugate flow,* it is the potential $\phi' = \psi$ that is many-valued in the manner just described; the stream function $\psi' = -\phi$ is single-valued. Consequently there is circulation of magnitude

$$\Gamma = -\Delta\phi' = 2\pi A$$

around any curve encircling $(c,0)$ once, or of magnitude $-2\pi A$ if the curve encircles $(-c,0)$ once, whereas the circulation vanishes around curves encircling both points the same number of times. A simple line vortex may be supposed to exist at $(c,0)$ and another of equal strength but opposite sign at $(-c,0)$; these are called a vortex pair. The components of velocity for this case are $u' = -v$, $v' = u$, where u and v are given by [41d,e]. There is flow symmetry with respect to the y-axis, geometrical symmetry with respect to x.

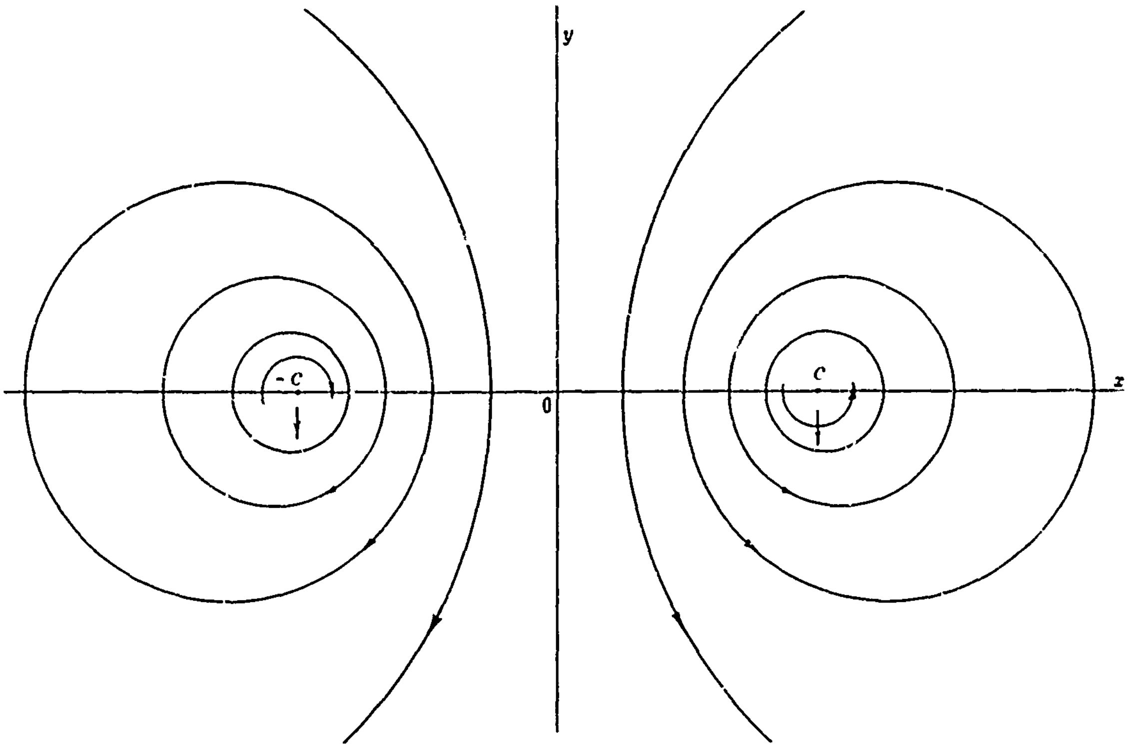


Figure 55 – Streamlines for a vortex pair. The arrows indicate the direction of motion of the vortices if they move with the fluid; see Section 41.

If the flow is assumed to be steady, the vortices are stationary. Alternatively, each vortex may be supposed to move with the average velocity of the fluid in its neighborhood, as a vortex does in a real fluid; in the present case, this velocity is simply that due to the other vortex, or, by [401], $U = A/2c = \Gamma/4\pi c$, directed toward negative y if A and Γ are positive. The pair of vortices thus advances without change of the distance between them; see Figure 55, where the direction of advance according to this assumption is shown by an arrow. The formulas will continue to represent the motion at each instant provided the axes are allowed to move with the vortices.

A pair of vortices *of the same sign* was discussed by Greenhill[26]. Trains of vortices were introduced by von Karman[27]; the streamlines for a typical Karman[27]; the streamlines for a typical Karman colum or "Karman street" are shown in Figure 56.

The transformation $w = A\,[\ln(z + ic) - \ln(z - ic)]$ represents the same flow rotated through 90 degree, with the source and sink or the vortices at $(0, \pm c)$.

It may be remarked also that toward infinity

$$\ln\frac{z+c}{z-c} = \ln\left(1+\frac{c}{z}\right) - \ln\left(1-\frac{c}{z}\right) = \frac{2c}{z} + \frac{2}{3}\frac{c^3}{z^3}\ldots$$

so that toward infinity the flow due to either a source and sink or a vortex pair approximates that of a dipole; compare Equation [37a].

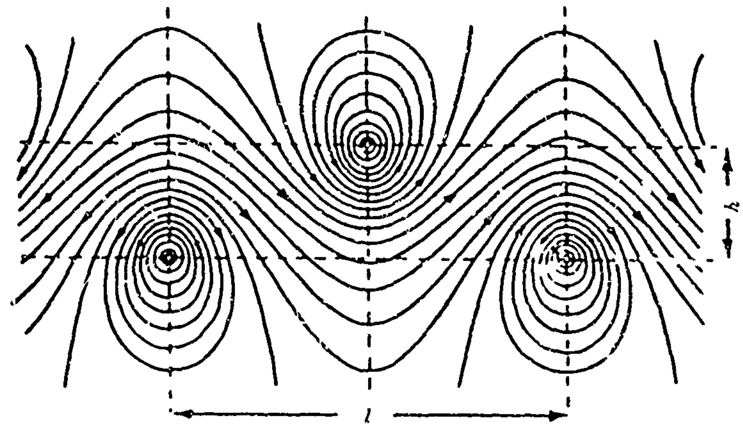


Figure 56 – Streamlines due to a Kármán vortex train or "street." (Copied from Reference 28.)

Further Geometrical Notes

Any given ϕ circle has a radius $R_\phi = c\, csch\, |(\phi/A)|$, and its center is at $x = c$ coth (ϕ/A). Thus the points $(\pm\, c,0)$ are inverse points with respect to each of these circles; for, the distances of the points from the center of any circle are $d_1 = |c - c \coth(\phi/A)|$ and $d_2 = |-c - c \coth(\phi/A)|$, and $d_1\, d_2 = c^2 [\coth^2 (\phi/A) - 1] = R_\phi^{\,2}$. The equation of the circle might be written, from [41i],

$$\ln \frac{r_1}{r_2} = \frac{\phi}{A} = \pm \sinh^{-1} \frac{c}{R_\phi} \qquad [41m]$$

the sign depending upon whether r_2 or r_1 is the greater.

The ψ arcs, on the other hand, have a radius $R_\psi = c\, csc\, |(\psi/A)|$ and are centered at $y = -c \cot(\psi/A)$. The region between any two of these arcs is mapped onto a strip of the w-plane lying between the corresponding values of ψ. The entire z-plane is thus mapped onto a strip of width $2\pi A$, between $\psi = -\pi A$ and $\psi = \pi A$, and it is mapped again on each successive strip of the same width.

The arcs for $\psi = -\pi A/2$ and $\psi = -3\pi A/2$ are semicircles which together form a circle of radius c centered at the origin and passing through the singular points at $(\pm\, c,0)$. Thus the transformation can also be used to transform the interior of a circle into an infinite strip. The strip is $\pi|A|$ wide and parallel to the real axis of ψ; it can be shifted so as to lie between the lines $\psi = \pm\, \pi A/2$ by adding $i\,\pi A$ to the value of w and hence πA to that of ψ, and, since $i\,\pi = \ln(-1)$, the transformation can then be written

$$w = A\,[\ln(c+z) - \ln(c-z)] \qquad [41n]$$

Here amp $(c-z)$ has been chosen so that $c - z = r_1\, e^{i(\theta_1 - \pi)}$ and $\ln(c-z) = \ln(z-c) - i\pi$, as shown in Figure 57; thus now $\psi = A\,(\theta_2 - \theta_1 + \pi)$. The quantity $\pi - \theta_1$ also equals the internal angle θ_1' between $c0$ and cz measured positively clockwise; in terms of this angle,

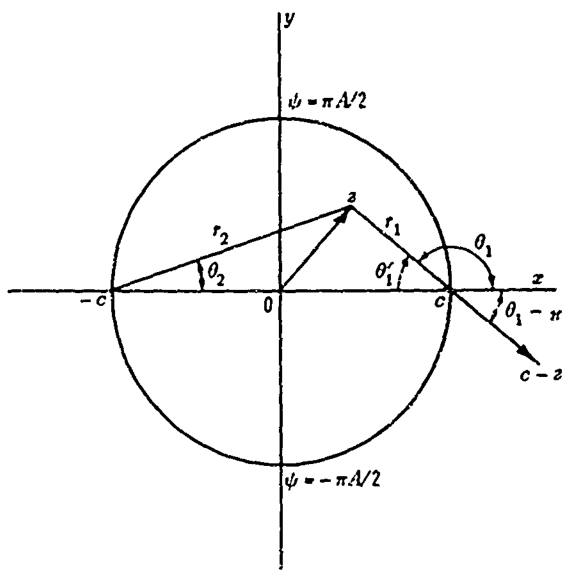


Figure 57 – Diagram to illustrate the transformation of the interior of a circle into an infinite strip, by Equation [41n].

$c - z = r_1\, e^{-i\theta_1'}$ and $\psi = A\,(\theta_1' + \theta_2)$.

The x-axis between $\pm c$ now transforms into the entire real axis or $\psi = 0$. The transformation can be visualized by imagining the interior of the circle to be drawn out into a strip as the points c and $-c$ recede in opposite directions to infinity.

(For notation and method; see Section 34; Reference 1, Article 64, 155; Reference 2, Sections 6.50, 8.22, 13.30.)

42. CIRCULATING FLOW: CYLINDERS, VORTICES, A WALL

The flow due to a vortex pair, as described in the last section, may be used to represent a circulating flow around parallel cylinders. Here the potential and stream function are $\phi' = \psi$, $\psi' = -\phi$, but it will be convenient to drop the primes. Then the new ϕ and ψ may be derived from the modified transformation

$$w = \phi + i\,\psi = iA\,[\ln(z-c) - \ln(z+c)] \qquad [42a]$$

whence, in the notation of the last section,

$$\phi = -A(\theta_1 - \theta_2), \quad \psi = -A\,\ln\frac{r_2}{r_1} \qquad [42b,c]$$

$$u = -A\left(\frac{\sin\theta_1}{r_1} - \frac{\sin\theta_2}{r_2}\right) = -\frac{4Acxy}{r_1^2\, r_2^2} \qquad [42d]$$

$$v = -A\left(\frac{\cos\theta_1}{r_1} - \frac{\cos\theta_2}{r_2}\right) = 2Ac\,\frac{x^2 - y^2 - c^2}{r_1^2\, r_2^2} \qquad [42e]$$

where

$$r_1 = [(x-c)^2 + y^2]^{1/2}, \quad r_2 = [(x+c)^2 + y^2]^{1/2}$$

and the angles θ_1 and θ_2 are shown in Figure 53. The value of q is again given by Equation [41h].

The equipotential curves are now the circular arcs ending at ($\pm c$,0), while the streamlines are the circles about these points; the equations of the arcs and circles, respectively, are, from [41k,l],

$$x^2 + \left(y + c \cot \frac{\phi}{A}\right)^2 = c^2 \csc^2 \frac{\phi}{A} = R_\phi^2 \qquad [42f]$$

$$\left(x + c \coth \frac{\psi}{A}\right)^2 + y^2 = c^2 \operatorname{csch}^2 \frac{\psi}{A} = R_\psi^2 \qquad [42g]$$

where R_ϕ, R_ψ denote the corresponding circular radii. The points ($\pm c$,0) are inverse points with respect to each circle; and the equation of any ψ circle can also be written in terms of geometrical quantities, from [41m], as

$$\ln \frac{r_2}{r_1} = \pm \sinh^{-1} \frac{c}{R_\psi} \qquad [42h]$$

If $A > 0$, ψ has everywhere the opposite sign to x, whereas ϕ is many-valued. If ϕ is assumed to be zero on the x-axis wherever $|x| > c$, it decreases in passing above the points ($\pm c$,0), becomes $-\pi A$ on the x-axis between these points, and decreases further to $-2\pi A$ on returning below the points ($\pm c$,0) to the starting point, where $|x| > c$. Thus there is circulation of magnitude $\Gamma = 2\pi A$ about (c,0) and of magnitude $-2\pi A$ about ($-c$,0).

By inserting cylindrical boundaries along one or two of the ψ circles, a number of cases of motion with circulation can be handled. In order to apply the formulas to a given case, it is necessary to find A and c, and the location of the origin of coordinates, in terms of given quantities.

A. Two Circular Cylinders

Two circular cylinders neither enclosing the other, with axes D distance apart, may be represented by causing two of the ψ circles to coincide with the circles representing the cylinders. For example, the circles may be those labeled 1 and 2 in Figure 54, where number 1 encloses the point (c,0) and number 2 the point ($-c$, 0). Let the given circulation around any curve encircling cylinder number 1 just once in the positive direction, but not cylinder number 2, be Γ, and that around any curve encircling number 2 only, $-\Gamma$. Let ψ have values ψ_1 and ψ_2 on the two cylinders, respectively. Draw the x-axis from 2 toward 1, as in Figure 54. Then $A = \Gamma/2\pi$.

In this case ψ_2 has the sign of A or Γ, and ψ_1 has the opposite sign; for, $r_2/r_1 < 1$ near cylinder 2 and $r_2/r_1 > 1$ near cylinder 1. If $A > 0$, ψ increases from 1 to 2, corresponding to a downward flow between the two cylinders. Hence, whatever the sign of A,

$$R_1 = -c \operatorname{csch} \frac{\psi_1}{A}, \quad R_2 = c \operatorname{csch} \frac{\psi_2}{A} \qquad [42h,i]$$

$$x_1 = -c \coth \frac{\psi_1}{A}, \quad x_2 = -c \coth \frac{\psi_2}{A} \qquad [42j,k]$$

or

$$x_1 = \sqrt{c^2 + R_1^2}, \quad x_2 = -\sqrt{c^2 + R_2^2} \qquad [42l,m]$$

Then $D = x_1 - x_2$, and, after eliminating all radicals by squaring twice,

$$4\, c^2 D^2 = \left[D^2 - (R_1 + R_2)^2\right]\left[D^2 - (R_1 - R_2)^2\right] \qquad [42n]$$

This formula fixes c when D, R_1, and R_2 are given, and the values of x_1 and x_2 then locate the origin of coordinates.

The singular points $(\pm c,0)$ lie inside the cylinders. Hence a valid representation is obtained of purely circulatory flow between and around two parallel cylinders. The difference $\psi_2 - \psi_1$ represents the volume of fluid that passes between them per second, per unit of their length.

B. Line Vortex Outside a Circular Cylinder

If cylinder number 1 is omitted and the formulas are continued down to the point $(c,0)$, the ideal flow is represented due to a line vortex outside a rigid cylindrical boundary of circular cross-section. The vortex, located at $(c,0)$, is at a distance h from the axis of the cylinder, which is located at $(x_2,0)$, where $h = c - x_2$. If R is written in place of R_2 for the radius of the cylinder and ψ_2 as before for the value of ψ on it, using [42i] and [42m] (Figure 58),

$$R = c \operatorname{csch} \frac{\psi_2}{A}, \quad h = c + \sqrt{c^2 + R^2}, \quad c = \frac{h^2 - R^2}{2h}, \quad x_2 = -\frac{h^2 + R^2}{2h} \qquad [42o,p,q,r]$$

The last equations serve to fix c and x_2 when h and R are given, and $A = \Gamma/2\pi$, where Γ is the assumed circulation around the vortex. The circulation around a curve encircling the cylinder once in a positive direction is $-\Gamma$.

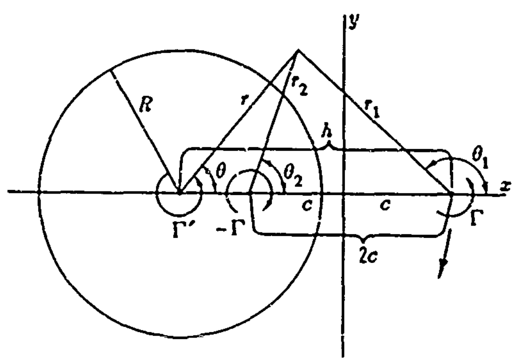


Figure 58 – Illustration of a line vortex of circulational strength Γ near a circular cylinder with circulation $\Gamma' - \Gamma$ around it; see Section 42B.

The term $-iA \ln(z+c)$ in the expression for w as given in [42a] represents the potential due to an image vortex of equal and opposite strength located at $(-c,0)$, or at a distance h' from the axis of the cylinder where $h' = -c - x_2 = \sqrt{c^2+R^2} - c$. Thus $hh' = R^2$, so that the vortex and its image are located on inverse lines with respect to the cylinder.

The stream function has the same sign as Γ or A near the cylinder, the opposite sign near the vortex.

If the vortex is assumed to move with the fluid, it revolves around the cylinder at the fixed distance h from its axis, in the opposite direction to that suggested by its own circulation and with a linear velocity equal to the fluid velocity caused by the image vortex in the cylinder, or with a velocity $A/2c$ or $\Gamma/4\pi c$. The formulas continue to represent the flow at each instant provided the axes rotate with the vortex, as does also the image vortex in the cylinder.

The circulation around the cylinder can be changed by superposing the flow due to another imaginary vortex located on the axis of the cylinder. Let the circulation due to this vortex be Γ'. The total circulation around the cylinder is then $\Gamma' - \Gamma$, and thus vanishes if $\Gamma' = \Gamma$. From [42b,c] and [40h,i], if r, θ are auxiliary polar coordinates with origin on the axis of the cylinder and coordinate axis parallel to x, as in Figure 58, potential and stream function for the resultant flow are:

$$\phi = -A(\theta_1 - \theta_2) - \frac{\Gamma'}{2\pi}\theta, \quad \psi = -A \ln\frac{r_2}{r_1} + \frac{\Gamma'}{2\pi}\ln r \qquad [42s,t]$$

The added components of velocity are $u' = -\Gamma' \sin\theta/(2\pi r)$, $v' = \Gamma' \cos\theta/(2\pi r)$. The added term in w is $(i/2\pi)\Gamma' \ln(z - x_2)$, where $z - x_2 = re^{i\theta}$.

If the original vortex is now assumed to move with the fluid, it revolves about the cylinder as before but at a linear velocity $\Gamma/(4\pi c) - \Gamma'/(4\pi h)$. The revolution is clockwise if $\Gamma' < h\Gamma/c$, otherwise counterclockwise; if $\Gamma' = h\Gamma/c$, the vortex is stationary.

C. A Cylinder of Radius R_2 enclosing one of Radius R_1

For this case, use is made of two circles lying on the same side of the origin of coordinates. Let the circulation around the inner cylinder, in the space between the two, be Γ. Let the x-axis be drawn in the direction from the axis of the inner cylinder, called number 1, toward the axis of the outer, called number 2; and let their axes be at x_1 and x_2, respectively. Let ψ have values ψ_1 and ψ_2 on the two cylinders, respectively. Then, using [42h] and [42l] for both cylinders,

$$R_1 = -c \operatorname{csch}\frac{\psi_1}{A}, \quad R_2 = -c \operatorname{csch}\frac{\psi_2}{A} > R_1 \qquad [42t,u]$$

$$x_1 = \sqrt{c^2 + R_1^2}, \quad x_2 = \sqrt{c^2 + R_2^2} > x_1 \qquad [42v,w]$$

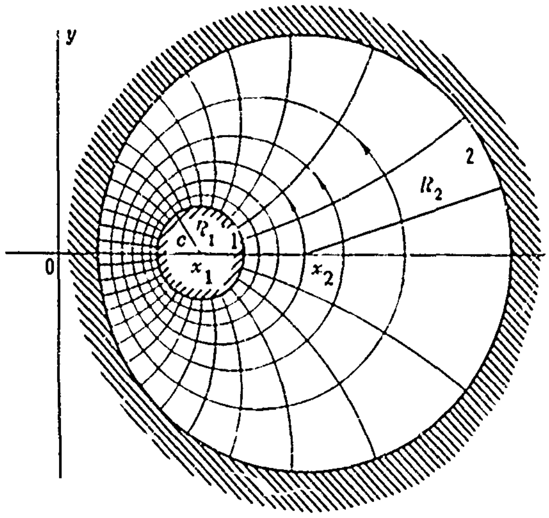


Figure 59 – Streamlines between two rigid cylindrical surfaces, centered at x_1, x_2. See Section 42 (C).

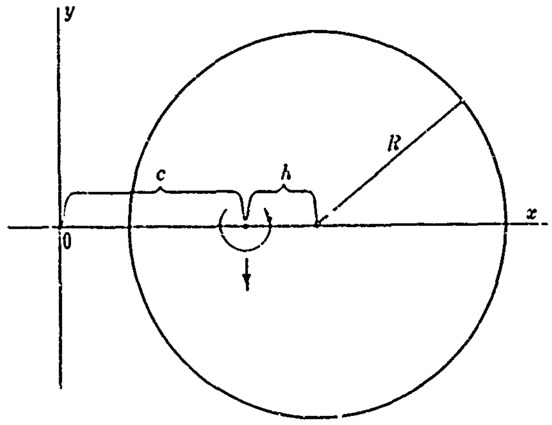


Figure 60 – Representation of a line vortex within a cylindrical shell. See Section 42 (D).

Here the distance D between the axes of the cylinders equals $x_2 - x_1$, but c is found to be given by [42n] as before, and $A = \Gamma/2\pi$. The value of x_1 or x_2 locates the origin of coordinates. In the region between the cylinders ψ and A have opposite signs.

The formulas represent circulating flow in the space between the two cylinders. A case is illustrated in Figure 59.

D. Line Vortex Inside a Cylinder

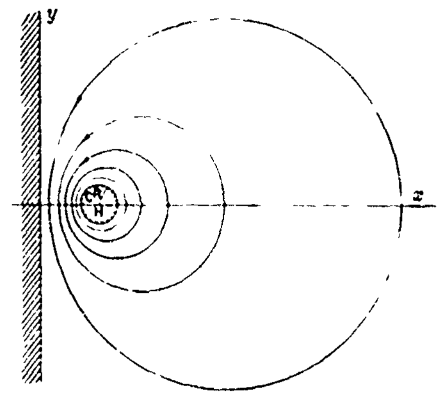
In the case just described, if the inner cylinder is omitted and the formulas are continued down to the point (c,0), the ideal flow is represented around a line vortex inside of a rigid cylindrical shell. The vortex is at (c,0) and there is circulation $\Gamma = 2\pi A$ about any closed curve lying inside the shell and encircling it once in the positive direction. If R is written for the radius of the shell, ψ_0 instead of ψ_2 for the value of ψ on it, and h for the distance of the vortex from the axis of the shell, which is at $(x_2,0)$, then $h = x_2 - c$ and from [42u] and [42w],

$$R = -c \operatorname{csch}\left(\frac{\psi_0}{A}\right), \quad h = \sqrt{c^2 + R^2} - c, \quad c = (R^2 - h^2)/2h \qquad [42x,y,z]$$

which fixes c when R and h are given. The origin lies outside of the shell, at a distance $h + c$ toward negative x from its axis, and inside the shell ψ has the opposite sign to A; see Figure 60.

If the vortex is assumed to move with the fluid, it revolves about the axis of the cylindrical shell at velocity $A/2c$, as in Case B; but here the direction of revolution is the same as that suggested by the circulation around the vortex. If it is located on the axis, the vortex is stationary.

Figure 61 – Streamlines between a circular cylinder and a wall. See Section 42 (E).



E. Cylinder and Plane Wall

By expanding the outer circle in Case C until it coincides with the y-axis, circulating flow is represented between a circular cylinder and a plane boundary or wall. Writing R instead of R_1 for the radius of the cylinder, H for the distance of its axis from the wall, and ψ_0 instead of ψ_1 for the value of ψ on it, it is found that $H = x_1$ and

$$R = -c \operatorname{csch}\left(\frac{\psi_0}{A}\right), \quad H = \sqrt{c^2+R^2}, \quad c = \sqrt{H^2-R^2} \qquad [42a', b', c']$$

which fixes c when R and H are given. The circulation around any closed curve encircling the cylinder positively once but not crossing the wall is Γ or $2\pi A$.

Since $\psi = 0$ on the wall, $-\psi_0$ represents the volume of fluid that passes between unit length of the cylinder and the wall per second, taken positive in the direction of counterclockwise motion around the cylinder. See Figure 61.

F. Line Vortex and Rigid Wall

If, in Case E, R is allowed to shrink to zero, the flow is represented due to an ideal line vortex parallel to a rigid wall and distant H or c from it. The wall is at $x = 0$; the vortex is at $(H,0)$, and the circulation around it is Γ or $2\pi A$. The terms in ϕ and ψ that involve reference to $(-H,0)$ can be regarded as arising from an image vortex at $(-H,0)$.

The velocity at the wall, from [42e], in which now

$$c = H, \quad r_1 = r_2 = (y^2+H^2)^{1/2}$$

is

$$v = -\frac{2AH}{y^2+H^2} = -\frac{\Gamma}{\pi}\frac{H}{y^2+H^2}$$

If the vortex is again assumed to move with the fluid, it moves parallel to the wall with the velocity due to the image vortex, which is $A/2H$ or $\Gamma/(4\pi H)$, toward negative y if $A > 0$. The axes must be assumed to move with the vortex.

The *forces on the cylinders* in any of the preceding cases, when the motion is steady, are most easily found from the Blasius theorem, which will be proved in Section 74. When only the two original vortices are present, from [42a]

$$\left(\frac{dw}{dz}\right)^2 = -A^2\left(\frac{1}{z-c} - \frac{1}{z+c}\right)^2$$

Substitution for $(dw/dz)^2$ in Equation [74g] gives, after a slight algebraic change,

$$X_1 - i\,Y_1 = -\frac{i}{2}\rho\,A^2 \oint \left[\frac{1}{(z-c)^2} - \frac{1}{c(z-c)} + \frac{1}{c(z+c)} + \frac{1}{(z+c)^2}\right] dz$$

where X_1 and Y_1 represent x and y components of the force per unit length on any cylinder due to fluid outside it, and the integral is to be taken around the circle representing the cylinder. The integral is easily evaluated by the method of residues as explained in Section 30. If the cylinder encloses the singular point $(-c,0)$ but not $(c,0)$,

$$\oint \frac{dz}{z+c} = 2\pi i$$

whereas all other terms of the integrand give zero. Thus

$$X_1 - i\,Y_1 = \frac{\pi\rho\,A^2}{c} = \frac{\rho\,\Gamma^2}{4\pi c}$$

where $\Gamma = 2\pi A$ and represents the circulation around the cylinder. Since $\rho\Gamma^2/4\pi c$ is real, $Y_1 = 0$, and the total force per unit length on the cylinder is

$$X_1 = \frac{\rho\Gamma^2}{4\pi c} \qquad [42d']$$

If the cylinder encloses the point (c,0) instead,

$$\oint \frac{dz}{(z-c)} = 2\pi i$$

and the sign of X_1 is reversed.

If another cylinder or a wall is present, as in Cases A, C, E, an equal and opposite force acts on it. The force on a wall is easily verified by direct integration of the Bernoulli term in the pressure.

If there is only an ideal line vortex at the point (c,0), the reactive force may be imagined to act on the vortex, but the formula for the force is correct only if the vortex is assumed to be stationary. If the fluid and stationary vortex are inside a cylindrical shell, the force on the shell is the same as if an inner cylinder were present with a circulation Γ around it equal to that around the vortex.

The direction of the force on a cylinder or on the wall is in all cases such as to draw it toward the other cylinder, or toward the wall or vortex, along the shortest path between them.

In the extended example case considered under Case B, where a vortex is near a cylinder having total circulation $\Gamma' - \Gamma$ around it, w contains another term and

$$\frac{dw}{dz} = \frac{iA}{z-c} - \frac{iA}{z+c} + \frac{i\Gamma'}{2\pi}\frac{1}{z-x_2}$$

The poles at $z = -c$ and $z = x_2$ both lie inside the cylinder and contribute to the integral. The product terms can be treated as before; for example,

$$\frac{2}{(z-c)(z-x_2)} = \frac{2}{c-x_2}\left(\frac{1}{z-c} - \frac{1}{z-x_2}\right)$$

The latter product, arising from two poles that lie inside the path of integration, gives zero in the integration, as is always the case with included poles. The force on the cylinder is thus found to be, using $A = \Gamma/2\pi$ where Γ is the circulation around the vortex,

$$X_1 = \frac{\rho\Gamma^2}{4\pi c} - \frac{\rho\Gamma\Gamma'}{2\pi h}$$

A positive value of X_1, means that the force acts toward the vortex.

Finally, in the case of a vortex moving freely parallel to a wall as described under Case F, the motion can be made steady without altering the force by imparting to everything a velocity equal and opposite to that with which the vortex is moving. The fluid velocity at the wall is then

$$v = \frac{\Gamma}{4\pi H} - \frac{\Gamma}{\pi}\frac{H}{y^2 + H^2}$$

On the assumption of zero pressure at infinity, the force on the wall is, from [34h], in which $U = \Gamma/4\pi H$ here,

$$\frac{1}{2}\rho\int\left[\left(\frac{\Gamma}{4\pi H}\right)^2 - v^2\right]dy = \frac{1}{2}\rho\left(\frac{\Gamma}{4\pi}\right)^2\int_{-\infty}^{\infty}\left(\frac{8}{y^2+H^2} - \frac{16H^2}{(y^2+H^2)^2}\right)dy = 0$$

To evaluate the second integral, put $y = H\tan\theta$.

(For notation and method; see Section 34; Reference 1, Articles 64, 155; Reference 2, Section 13.30, 13.31, 13.40, 13.41; for line vortex and cylinder, Müller[28] and Morris.[29]

43. LINE SOURCE AND PLANE WALL

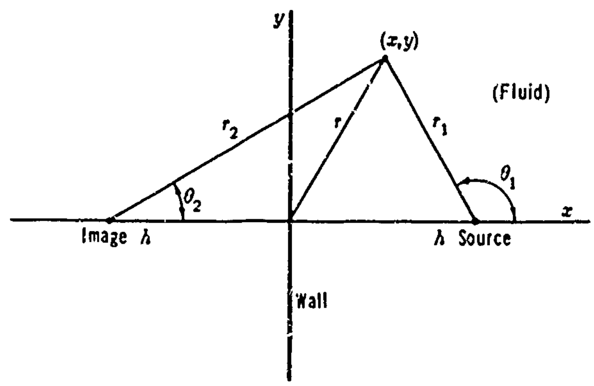
$$w = -A\,[\ln(z+h) + \ln(z-h)] \qquad [43a]$$

A and h real constants.

Some problems are easily solved by superposing known types of flow.

Consider for example, a uniform line source in fluid that is bounded, at a distance h from the source, by a plane rigid wall parallel to the source, as in Figure 62. If the flow is

Figure 62 – A line source near a wall.



first assumed to be the same as it would be in unbounded fluid, the velocity has a component normal to the wall. Upon this flow let there be superposed the flow that would be associated in an infinite mass of fluid with an equal and parallel line source located on the opposite side of the wall, at a distance h from it and on the perpendicular from the given source to the wall produced. Then at the wall the normal components of these two flows will cancel and the boundary condition will thus be satisfied. These two partial flows are assumed to exist only on the side of the wall on which the original source lies. The imaginary second source is called an image of the first in the plane of the wall.

Let the x-axis be drawn through the source and its image, being thus perpendicular to the wall, and the y-axis along the wall. Then, from Section 40, the complex potential is as given above and the resultant potential ϕ and streamfunction ψ are

$$\phi = -A \ln (r_1 r_2), \quad \psi = -A(\theta_1 + \theta_2) \qquad [43b,c]$$

the significance of $r_1, r_2, \theta_1, \theta_2$ is exhibited in Figure 62. If it is desired to balance the equations dimensionally, $r_1 r_2$ can be replaced by $r_1 r_2/a^2$ in ϕ, thereby merely adding a constant to all values of ϕ. The volume of fluid emitted by unit length of the line source per second is $2\pi A$.

Some of the streamlines are shown in Figure 63, above the x-axis only, relative to which the flow is symmetrical. The source is at S. The streamlines are arcs of rectangular hyperbolas with centers at the origin 0, given by

$$x^2 + 2xy \cot \frac{\psi}{A} - y^2 = h^2 \qquad [43d]$$

as is easily verified by writing out $\tan(\theta_1 + \theta_2)$. The equipotential curves are Bernoullian lemniscates.

Since $z = x + iy$

$$q^2 = \left|\frac{dw}{dz}\right|^2 = 4A^2 \frac{|z|^2}{|z^2 - h^2|^2} = 4A^2 \frac{x^2 + y^2}{(x^2 - y^2 - h^2)^2 + 4x^2 y^2}$$

whence

$$q = \frac{2Ar}{r_1 r_2} \qquad [43e]$$

where

$$r^2 = x^2 + y^2,\ r_1^2 = (x-h)^2 + y^2,\ r_2^2 = (x+h)^2 + y^2$$

Thus at infinity $q \to 2A/r$, as in the flow due to a single source of double strength at 0. On the wall $x = 0$, $r = |y|$ and $q = 2A\ |y|/(h^2 + y^2)$; 0 is a stagnation point.

The components of velocity are, from Equation [43b],

$$u = A\left(\frac{\cos\theta_1}{r_1} + \frac{\cos\theta_2}{r_2}\right),\ v = A\left(\frac{\sin\theta_1}{r_1} + \frac{\sin\theta_2}{r_2}\right) \qquad [43f,g]$$

The net force per unit length on the wall due to the Bernoulli term in the pressure is

$$F = \frac{1}{2}\rho\int_{-\infty}^{\infty} q^2\,dy = \frac{1}{2}\rho\int_{-\infty}^{\infty}\frac{4A^2y^2}{(h^2+y^2)^2}\,dy = \frac{\pi\rho A^2}{h} \qquad [43h]$$

The source can be said to attract the wall.

[For notation and method; see Section 34; Reference 2, Sections 8.31, 8.41; also Reference 5]

44. ROW OF EQUAL SOURCES OR VORTICES; SOURCE MIDWAY BETWEEN WALLS OR ON ONE WALL; CONTRACTED CHANNEL

$$w = -A\ln\sinh\frac{\pi z}{a},\ a \text{ and } A \text{ real}, \qquad [44a]$$

$$\phi = -\frac{1}{2}A\ln\left(\cosh\frac{2\pi x}{a} - \cos\frac{2\pi y}{a}\right) \qquad [44b]$$

$$\psi = -A\tan^{-1}\left(\tan\frac{\pi y}{a}\Big/\tanh\frac{\pi x}{a}\right) \qquad [44c]$$

from hyperbolic formulas in Section 32 and $\ln(re^{i\theta}) = \ln r + i\,\theta$. In ϕ a constant term is dropped.

In ψ, $\tan^{-1}$ is to be interpreted so as to vary continuously with x and y. Then $\tan^{-1}$ and y vary in the same sense if x is held fixed at a positive value, whereas they vary in opposite directions if x is kept negative. While x remains positive, $\tan^{-1}$ may be assumed to vanish with y; then, continuity being assumed, either $\tan^{-1}$ and $\pi\,y/a$ are both positive angles in the first quadrant, or they are both negative angles in the fourth. The effect of letting x become negative is easily seen if $\tan(\pi\,y/2)$ is kept finite and not zero. For example, let both $\tan^{-1}$ and $\pi y/2$ be positive and in the first quadrant, with $x = x_1 > 0$. Then, if x is decreased to $-x_1$ without change in y, $\tan^{-1}$ increases to $\pi - \tan^{-1}[\tan(\pi y/a)/\tanh(\pi x_1/a)]$; whereas,

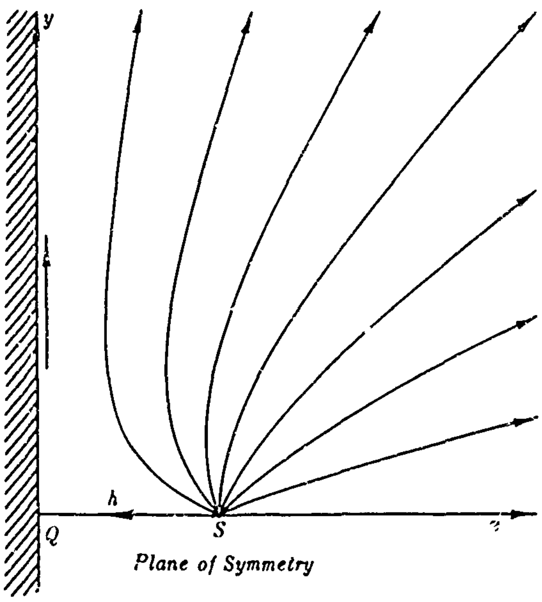


Figure 63 – Streamlines due to a line source S near a wall.

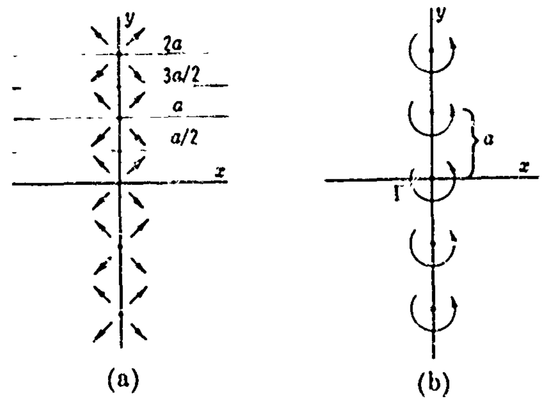


Figure 64 – A row of equal line sources or vortices.

if $\pi y/a$ and $\tan^{-1}$ are both negative and in the fourth quadrant, $\tan^{-1}$ decreases with decreasing x to $-\pi + \tan^{-1}[\tan(\pi y/a)/\tanh(\pi x_1/a)]$ at $x = -x_1$.

Hence,

$$u = \frac{\pi A}{aH}\sinh\frac{2\pi x}{a},\quad v = \frac{\pi A}{aH}\sin\frac{2\pi y}{a} \qquad [44d,e]$$

$$q^2 = \frac{\pi^2 A^2}{a^2 H}\left(\cosh\frac{2\pi x}{a} + \cos\frac{2\pi y}{a}\right),\quad H = \cosh\frac{2\pi x}{a} - \cos\frac{2\pi y}{a} \qquad [44f,g]$$

The expressions for ϕ, u, v, and q are periodic in the y direction with a period equal to a. The y-axis and the lines $y = 0, \pm a/2, \pm a, \pm 3/2\,a, \pm 2a, \ldots\ldots$ all represent planes of flow symmetry.

The flow is that due to equal line sources spaced a distance a apart along the y-axis; at each of the points $(0,0)$, $(0, \pm a)$, $(0, \pm 2a)$, etc., there occurs a line source emitting a volume $2\pi A$ per second per unit length. For, as the origin, for example, is encircled positively, $\tan^{-1}$ increases by 2π and ψ decreases by $2\pi A$. If $A < 0$, line sinks occur at these points. See Section 40 and Figure 64a.

In the corresponding *conjugate flow*, with potential $\phi' = \psi$ and stream function $\psi' = -\phi$, the sources are replaced by line vortices with circulation $\Gamma = 2\pi A$ about each. The velocity components are $u' = -v$, $v' = u$. At large distances from the row of vortices $u' \to 0$, $v' \to \pi A/a = \Gamma/2a$, since $\sinh(2\pi x/a)/\cosh(2\pi x/a) \to \pm 1$. See Figure 64b. Thus, if a large width h of the plane containing the vortices is encircled by a path, the circulation about this path is $2h\Gamma/2a = h\Gamma/a$, in agreement with the fact that h/a vortices are encircled. The substitution

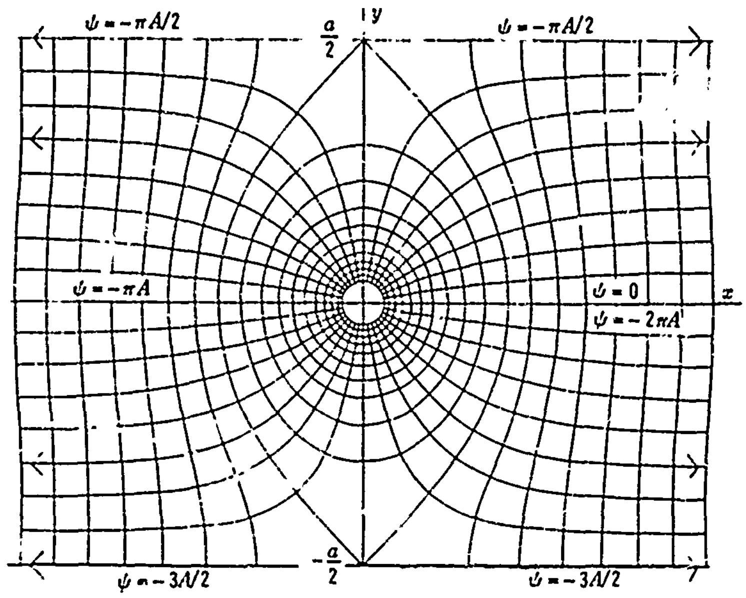


Figure 65 – Flow net due to a line source between walls, or forming one member of a row of sources; the source mentioned is at the origin. See Section 44. (Copied from Reference 253.)

sin $\pi z/a$ for sinh $\pi z/a$ rotates everything through 90 deg, with the sources or vortices on the x-axis.

A number of interesting physical cases can be constructed by the suitable insertion of walls.

(A) Line Source Between Parallel Walls

According to Equation [44c], ψ is constant along each of the lines $y = \pm\, a/2$. (Note that tan 90 deg = $\pm\,\infty$, tan 270 deg = $\pm\,\infty$.) Rigid walls may be inserted along these lines, and the formulas then represent flow due to a line source midway between two walls separated by a distance a. The flow net is shown in Figure 65. As indicated on the figure, a consistent set of values of ψ, which is many-valued, is as follows: $\psi = 0$ on the positive x-axis; $\psi = -\pi A/2$ along the positive y-axis to $y = a/2$ and then in both directions along the wall that lies at $y = a/2$; $\psi = -\pi A$ on the negative x-axis; $\psi = -3\pi A/2$ on the negative y-axis and along the other wall at $y = -a/2$; $\psi \to -2\pi A$ just below the x-axis.

As $x \to \pm\,\infty$, $v \to 0$, $u \to (\pi A/a) \tanh (2\pi x/a) \to \pm\,\pi A/a$. Thus the flow becomes uniform, and the total rate of outflow from the source is $2(a\,|u|) = 2\pi A$, as found before. In general, on the x-axis $v = 0$ and

$$u = \frac{\pi A}{a} \frac{\sinh\left(2\pi \frac{x}{a}\right)}{\cosh\left(2\pi \frac{x}{a}\right) - 1} = \frac{\pi A}{a} \coth \frac{\pi x}{a} \qquad [44h]$$

On the y-axis $u = 0$ and

$$v = \frac{\pi A}{a} \cot \frac{\pi y}{a} \qquad [44i]$$

On the walls at $y = \pm a/2$, $v = 0$ and

$$u = \frac{\pi A}{a} \tanh \frac{\pi x}{a} \qquad [44j]$$

(B) Line Source in a Stream Between Parallel Walls

If a uniform flow parallel to the walls is superposed, the resulting formulas can be written, in terms of real constants U and a real positive constant g:

$$\phi = U\left\{x + \frac{g}{2\pi}\left[\frac{\pi x}{a} - \frac{1}{2}\ln\left(\cosh\frac{2\pi x}{a} - \cos\frac{2\pi y}{a}\right)\right]\right\} \qquad [44k]$$

$$\psi = U\left\{y + \frac{g}{2\pi}\left[\frac{\pi y}{a} - \tan^{-1}\left(\tan\frac{\pi y}{a}/\tanh\frac{\pi x}{a}\right)\right]\right\} \qquad [44l]$$

$$u = U\left[-1 + \frac{g}{2a}\left(\frac{1}{H}\sinh\frac{2\pi x}{a} - 1\right)\right], \quad v = \frac{gU}{2aH}\sin\frac{2\pi y}{a} \qquad [44m,n]$$

The uniform velocity that is here superposed is $U(1+g/2a)$, but the resultant stream velocity at $x=+\infty$ is U, since, as $x\to\infty$, $u\to -U$. As $x\to -\infty$, $u\to -U\,(1 + g/a)$. On the walls at $y=\pm a/2$,

$$u = U\left[-1 + \frac{g}{2a}\left(\tanh\frac{\pi x}{a} - 1\right)\right] \qquad [44o]$$

On the x-axis

$$u = U\left[-1 + \frac{g}{2a}\left(\coth\frac{\pi x}{a} - 1\right)\right] \qquad [44p]$$

and a stagnation line occurs at $x = x_Q$ where

$$x_Q = \frac{a}{\pi}\tanh^{-1}\frac{g}{2a+g} \qquad [44q]$$

With a suitable definition of $\tan^{-1}$, $\psi = 0$ occurs on the x-axis and also on the dividing surface S defined by

$$\left(1 + \frac{2a}{g}\right)\frac{\pi y}{a} = \tan^{-1}\left(\tan\frac{\pi y}{a} / \tanh\frac{\pi x}{a}\right) \qquad [44r]$$

This surface passes through the point $(x_a, 0)$, and toward $x = -\infty$ its sides become parallel and a distance L apart where L represents twice the limiting numerical value of y or

$$L = \frac{ga}{a+g} \qquad [44s]$$

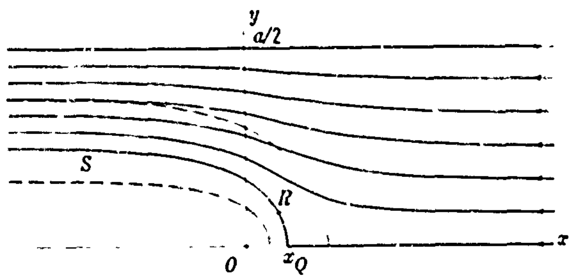


Figure 66 – Flow past a semi-infinite cylinder between walls a apart, or along a channel narrowed in a certain manner. Constructed with use of a line source at 0. See Section 44.

Here, for continuity, $\tan^{-1}[-\tan(\pi L/2a)]$ is interpreted as $\pi - \pi L/2a$. On the walls, $\psi = aU/2$ at $y = a/2$, and $\psi = -aU/2$ at $y = -a/2$.

A semi-infinite cylinder can be inserted along S, and the flow is then represented between the walls and past this cylinder. The fluid approaches from $x = +\infty$ at velocity U and leaves, at $x = -\infty$, at velocity $U' = U(1 + g/a) = aU/(a-L)$.

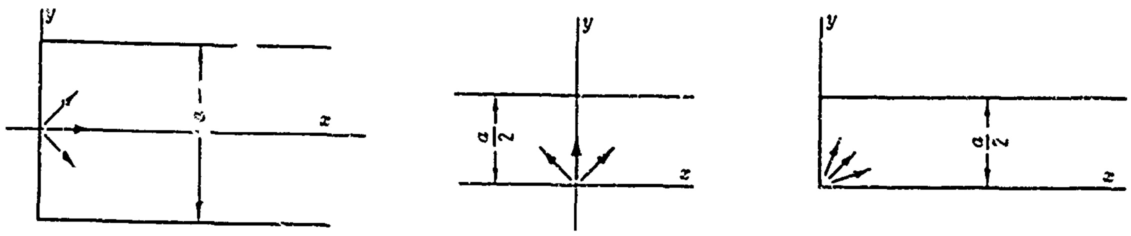
Streamlines for $g/a = 1$ are shown in Figure 66, for $y > 0$ only, since the x-axis represents a plane of symmetry. On the cylinder, $q = U$ at P, where $x = 0.087a$. Other possible forms of the cylinder, for $g/a = 1/2$ and $g/a = 3$, are shown by broken curves.

A wall could also be inserted along the positive x-axis up to x_Q and then along S, forming, with a wall at $y = a/2$, a channel narrowed in a certain manner.

If the sign of U is changed, all velocities are reversed without change in the geometrical flow net or the pressure. The diagram can be most easily reversed from left to right by reversing the positive direction for x.

(C) Line Source in One Wall or Corner of a Channel

In Case A, an additional wall can be inserted along the y-axis. It is then convenient to take $\psi = -\pi A/2$ for $y > 0$ and $\psi = \pi A/2$ for $y < 0$, so that ψ is continuous on the positive x-axis. The formulas of Case A, and the right-hand half of Figure 65, will then represent the flow in a semi-infinite channel with plane walls a apart, due to a line source along the middle of its base. See Figure 67a. The source may represent approximately fluid entering or leaving through a narrow slit; a volume πA passes through the slit per second, between any two planes of flow unit distance apart.



(a) (b) (c)

Figure 67 – Line source on a rectilinear boundary or in a corner. See Section 44 (C).

Or, alternatively, walls may be inserted along $y = 0$ and $y = a/2$ only. Then the line source, of the same strength, lies on a side extending to infinity in both directions, as in Figure 67b, and the streamlines are illustrated by the upper half of Figure 65.

Finally, walls may be inserted again along $y = 0$ and $y = a/2$, and also along $y = 0$. Then the formulas and one quarter of Figure 65 represent the flow due to a source or sink in one corner of a rectangular channel of width $a/2$, as in Figure 67c. The source emits $\pi A/2$ units of volume into the channel per unit time, between planes of flow unit distance apart.

(D) Channel with a Smooth Contraction

The mathematical formulas can be extended upward to $y = a$ without passing any singularity. The diagram between $y = 0$ and $y = a$ is then symmetric with respect to the line $y = a/2$. Let a uniform stream be superposed, as in Case B, and let curved infinite walls be inserted along any two streamlines lying between $y = 0$ and $y = a$. Then the diagram between these lines represents flow along a two-dimensional channel whose cross-section changes smoothly at a certain place along its length. See Figure 68, where any two curves may represent the walls.

Equations [44k] to [44p] are applicable. Assume $\psi = 0$ and $\tan^{-1} [(\tan \pi y/a)/\tanh (\pi x/a)] = 0$ on the x-axis. Then, for continuity, $\tan^{-1}$ passes from the first quadrant into the second as y increases past $y = a/2$, provided $x > 0$. but, if $x < 0$, these two quadrants are interchanged. Thus, as $x \to +\infty$ and $\tanh (\pi x/a) \to 1$, $\tan^{-1} [(\tan (\pi y/a)/\tanh (\pi x/a)] \to \pi y/a$; whereas, as $x \to -\infty$, $\tan^{-1} \to \pi - (\pi y/a)$. Hence

$$\text{as } x \to +\infty, \quad \psi \to Uy;$$
$$\text{as } x \to -\infty, \quad \psi \to U\left[\left(1+\frac{g}{a}\right)y - \frac{g}{2}\right]$$

Thus, if ψ_1 and ψ_2 are the values of ψ on the curved walls, their distance apart, given by the difference in y, is

$$L_1 = \frac{a}{a+g}\,\frac{\psi_2-\psi_1}{U} \text{ at } x = -\infty, \quad L_2 = \frac{\psi_2-\psi_1}{U} \text{ at } x = +\infty$$

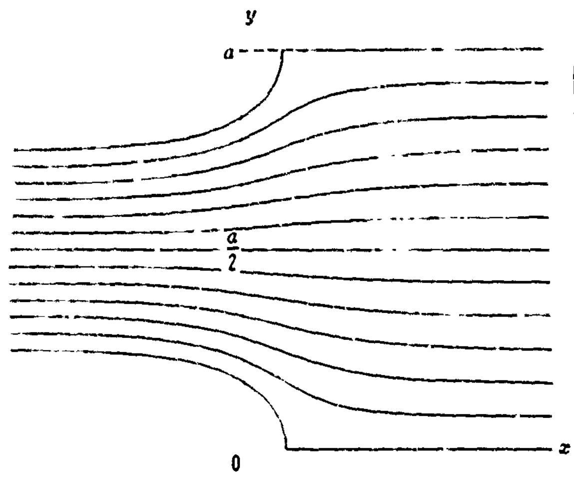
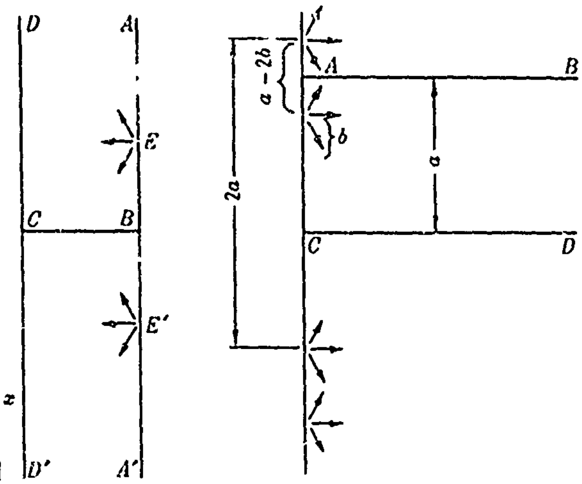


Figure 68 – Streamlines in a channel narrowed in a certain manner, as constructed with use a line sources. See Section 44(D).



(a) (b)

Figure 69 – Line source on a rectilinear boundary. See Section 44(E).

The ratio of contraction or L_1/L_2 is in all cases $a/(a+g)$.

If the x-axis is drawn along the central line of symmetry or $y = a/2$, then in [44k], [44l], and [44m,n] $\tan^{-1}$ is replaced by $\cot^{-1}$, tan by cot, sin by − sin, and cos by − cos.

If infinite plane walls are inserted along $y = 0$ and $y = a$, they form a straight channel with a source in each wall.

(E) Source Anywhere on Wall of a Rectangular Vessel

More general cases can be constructed by combining two flows of the type described in this section. Two examples may be noted.

To represent approximately flow into or out of a small slit at E in the side of a rectangular two-dimensional vessel ABCD, as sketched in Figure 69a, flows may be superposed due to two equal rows of sources or sinks perpendicular to the side AA′ of an infinite channel, with sources or sinks at E and E' on this side, and a partition may then be inserted along the plane of symmetry BC.

To represent approximately flow through a small slit located in the bottom of a rectangular vessel ABCD but displaced a distance ℓ from the center, combine two rows of equal sources, with the sources 2a apart in each row but the rows differing in position by a − 2b, where a is the width of the vessel, and insert partitions along two consecutive planes of symmetry, such as AB and CD in Figure 69b.

In both cases, of course, only the flow inside the vessel is represented. (For notation and method; see Section 34; Reference 2, Article 10.4, 10.5; Jaffé (30); Cisotti (31).

45. ALTERNATING VORTICES OR SOURCES; VORTEX MIDWAY BETWEEN WALLS

$$w = \frac{i\Gamma}{2\pi} \ln \tanh \frac{\pi z}{2a}\ , \ a \text{ and } \Gamma \text{ real} \qquad [45a]$$

$$\phi = -\frac{\Gamma}{2\pi} \tan^{-1}\left[\sin\frac{\pi y}{a}\Big/\sinh\frac{\pi x}{a}\right],\ \psi = -\frac{\Gamma}{4\pi} \ln \frac{\cosh\frac{\pi x}{a} + \cos\frac{\pi y}{a}}{\cosh\frac{\pi x}{a} - \cos\frac{\pi y}{a}} \qquad [45b,c]$$

$$u = -\frac{\Gamma}{2aG} \cosh\frac{\pi x}{a} \sin\frac{\pi y}{a},\quad v = \frac{\Gamma}{2aG} \sinh\frac{\pi x}{a} \cos\frac{\pi y}{a} \qquad [45d,e]$$

$$q = \frac{\Gamma}{2aG},\quad G = \sinh^2\frac{\pi x}{a} + \sin^2\frac{\pi y}{a} \qquad [45f,g]$$

For the interpretation of $\tan^{-1}$, see Section 44.

On the x-axis $u = 0$, $\quad v = \Gamma\Big/\left(2a \sinh\frac{\pi x}{a}\right)$ [45h]

On the y-axis, $v = 0$, $\quad u = -\Gamma\Big/\left(2a \sin\frac{\pi y}{a}\right)$ [45i]

On the lines $y = \pm a/2$, $v = 0$, $u = \mp\Gamma\Big/\left(2a \cosh\frac{\pi x}{a}\right)$ [45j]

The $\tan^{-1}$ is to be defined so as to vary continuously with x and y; let one of its values on the positive x-axis be zero.

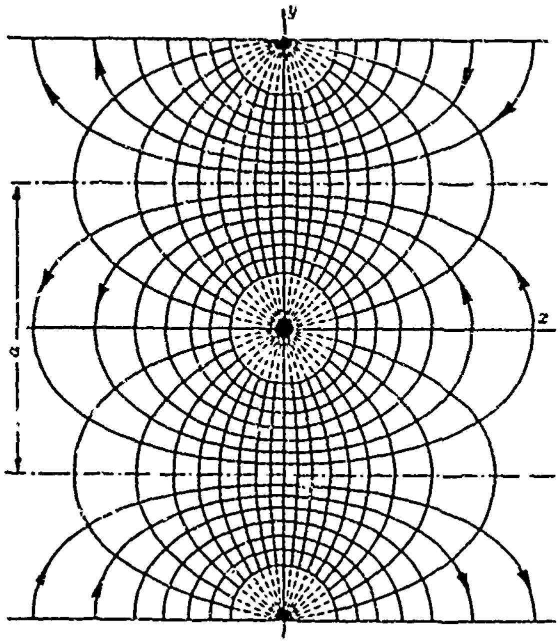
There is circulation Γ around the origin, since $\tan^{-1}$ increases by 2π and ϕ decreases by Γ as the origin is encircled. Furthermore, if y is increased by a, all functions merely change sign; hence there is circulation $-\Gamma$ about (0,a). Finally, everything repeats when y is changed by 2a. Thus the formulas represent the flow due to a row of vortices spaced a apart along the y-axis, all of equal strength but alternating in direction of rotation. Part of the flow net is shown in Figure 70.

The conjugate flow with $\phi' = \psi$, $\psi' = -\phi$ represents a row of alternate sources and sinks of equal strength, spaced a apart along the y-axis.

Parallel plane walls may be inserted along the lines $y = a/2$, on which $\psi = 0$. Streamlines for this case are shown by the central section of Figure 70.

The substitution of $\tan(\pi z/2a)$ for $\tanh(\pi z/2a)$ rotates everything through 90 degrees, with the sources or vortices on the x-axis. (For notation and method; see Section 34; Jaffé[30], Reference 32.)

Figure 70 – Flow net due to a row of vortices or sources of alternating sign, or due to a vortex between walls a apart.
(Copied from Reference 253.)



46. ROW OF EQUAL LINE DIPOLES ON A TRANSVERSE AXIS; DIPOLE MIDWAY BETWEEN WALLS, WITH PARALLEL AXIS; FLOW PAST CYLINDER BETWEEN WALLS OR THROUGH A GRATING

$$w = B \coth \frac{\pi z}{a}, \quad a \text{ and } B \text{ real;} \qquad [46a]$$

$$\phi = \frac{B}{H} \sinh \frac{2\pi x}{a}, \quad \psi = -\frac{B}{H} \sin \frac{2\pi y}{a} \qquad [46b,c]$$

$$u = \frac{2\pi B}{aH^2}\left(\cosh \frac{2\pi x}{a} \cos \frac{2\pi y}{a} - 1\right) \quad v = \frac{2\pi B}{aH^2} \sinh \frac{2\pi x}{a} \sin \frac{2\pi y}{a} \qquad [46d,e]$$

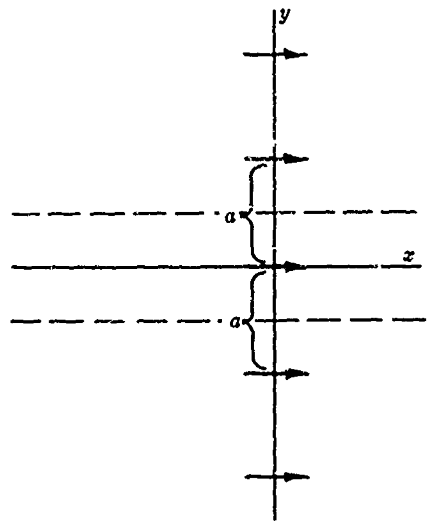
$$H = 2\left[\sinh^2 \frac{\pi x}{a} + \sin^2 \frac{\pi y}{a}\right] = \cosh \frac{2\pi x}{a} - \cos \frac{2\pi y}{a} \qquad [46f]$$

All functions are periodic in the y direction with a period a. The lines $y = 0, \pm a/2, \pm a, \ldots$ represent planes of flow symmetry; the y-axis represents a plane of geometrical symmetry.

At each of the points (0,0), $(0 \pm a)$, $(0 \pm 2a)$ . . . . on the y-axis there occurs a line dipole with line-dipole moment equal to aB/π and with its axis directed toward positive x (if $B > 0$). For, near the origin, for example, $w = B \cosh(\pi z/a)/\sinh(\pi z/a) \to aB/\pi z$ by use of the series [33f,g]; this represents a dipole, as in Section 37; see Figure 71.

In the *conjugate flow* the axes of the dipoles are directed along the y-axis; this is the flow of Section 47 rotated through 90 degrees.

Figure 71 – Row of equal line dipoles with transverse axes.



(1) Dipole Between Walls

On the lines $y = \pm a/2$, $\psi = 0$. Walls may be inserted along these lines, which are drawn broken in Figure 71. Then the flow is represented due to a dipole placed midway between parallel walls separated by a distance a; the axis of the dipole is parallel to the walls.

On the walls at $y = \pm a/2$, $v = 0$, $u = -\dfrac{2\pi B}{a}\left(\cosh\dfrac{2\pi x}{a} + 1\right)^{-1}$ [46g]

On the x-axis, $v = 0$, $u = \dfrac{2\pi B}{a}\left(\cosh\dfrac{2\pi x}{a} - 1\right)^{-1}$ [46h]

On the y-axis, $v = 0$, $u = -\dfrac{2\pi B}{a}\left(1 - \cos\dfrac{2\pi y}{a}\right)^{-1}$ [46i]

(2) Dipole in a Stream Between Walls

If a uniform stream at velocity U toward negative x is superposed, terms Uz, Ux, Uy are added in w, ϕ, ψ, respectively, and a term $-U$ in u. Assume that $B/U > 0$. Then $\psi = 0$, not only on the x-axis or median plane between the walls, but also on an oval cylindrical surface S whose equation is

$$Uy = \frac{B}{2}\sin\frac{2\pi y}{a} \Big/ \left(\sinh^2\frac{\pi x}{a} + \sin^2\frac{\pi y}{a}\right) \qquad [46j]$$

See Figure 72. The semidiameters of S, r_1, and r_2, in the x and y directions, are given by

$$\sinh\frac{\pi r_1}{a} = \sqrt{\frac{\pi B}{aU}}, \quad r_2 = \frac{B}{U}\cot\frac{\pi r_2}{a} \qquad [46k,l]$$

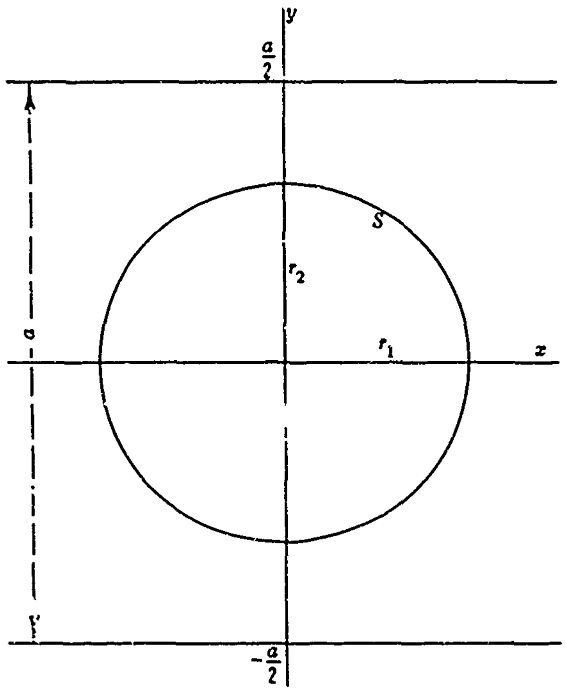


Figure 72 – Nearly circular cylinder between walls, obtained with use of dipoles in a stream. See Section 46.

To obtain the first of these equations, divide (46j) through by y, then let $y \to 0$, noting that $\sin(2\pi y/a)/y \to 2\pi/a$, and solve for $x = r_1$. The second equation has a root $0 < r_2 < a/2$ for any $B/U > 0$.

A cylinder may be inserted along the surface S. Then, if walls are also inserted along the lines $y = \pm a/2$, the formulas, modified as stated, represent a stream that flows between the walls and past this cylinder.

Equation [46k] may be solved for r_1 by using the series [331] for $\sinh^{-1}$:

$$\frac{\pi r_1}{a} = \left(\frac{\pi B}{aU}\right)^{1/2} - \frac{1}{6}\left(\frac{\pi B}{aU}\right)^{3/2} + \frac{3}{40}\left(\frac{\pi B}{aU}\right)^{5/2} \ldots\ldots$$

or

$$r_1 = r_0\left[1 - \frac{1}{6}\left(\frac{\pi r_0}{a}\right)^2 + \frac{3}{40}\left(\frac{\pi r_0}{a}\right)^4 \ldots\ldots\right],\quad r_0 = \sqrt{\frac{aB}{\pi U}} \qquad [46m,n]$$

To obtain a similar series for r_2, assume that

$$r_2 = r_0\left[1 + c_1\left(\frac{\pi r_0}{a}\right)^2 + c_2\left(\frac{\pi r_0}{a}\right)^4 \ldots\ldots\right]$$

write [46l] in the form

$$\frac{\pi^2 r_2^2}{a^2} = \frac{\pi B}{aU} \frac{\pi r_2}{a} \cot \frac{\pi r_2}{a}$$

use series [33e] for cot, substitute for r_2 in terms of r_0, and determine c_1, c_2 . . . . . by equating coefficients of like powers. The result is

$$r_2 = r_0 \left[1 - \frac{1}{6} \left(\frac{\pi r_0}{a} \right)^2 + \frac{11}{360} \left(\frac{\pi r_0}{a} \right)^4 \; . \; . \; . \; . \; . \right] \qquad [46o]$$

Thus, when $aB/\pi U$ is small, so that $\pi r_0/a$ is small, so that $\pi r_0/a$ is small, r_1 and r_2 agree to the second order, and S closely approximates a circle. Its radius becomes r_0 as $aB/U \to 0$. Even if $r_2 = a/4$, r_1 exceeds r_2 by less than 2 percent, although both are about 10 percent smaller than r_0.

The kinetic energy of the fluid, when the cylinder moves at velocity U in translation parallel to the walls while the fluid at infinity is at rest, is easily found from Equation [76a] in Section 76. For such motion the complex potential, obtained by dropping again the term Uz, is w as given by [46a]. Let the path of integration be displaced outward from the contour of the cylinder until it becomes a long rectangle with sides lying along the walls and ends at $x = \pm l$. Then on the walls $dz = dx$ and $\psi = 0$; hence the walls contribute nothing to (I') $\oint w \, dz$. On the ends, $dz = i\, dy$ and $\phi \to \pm B \sinh (2\pi l/a)/[\cosh (2\pi l/a) + 1] = \pm B \tanh (\pi l/a) \to \pm B$ as $l \to \infty$, so that the ends contribute

$$2B \int_{-a/2}^{a/2} dy = 2aB$$

Thus the kinetic energy of the fluid, per unit length of the cylinder, is, from [76a],

$$T_1 = \frac{1}{2} \rho \, U \, (2aB - US) = \frac{1}{2} \rho \, U^2 \left(2\pi r_0^2 - S \right) \qquad [46p]$$

where S is the cross-sectional area of the cylinder. If r_0/a is small, the radius of the cylinder can be taken to be, from [46m,o], $r = r_0 \, [1 - (\pi r_0/a)^2/6]$, so that $r_0 = r \, [1 + (\pi r/a)^2/6]$, approximately; and $S = \pi r^2$. Then (see Taylor[33])

$$T_1 = \frac{1}{2} \rho \, \pi r^2 \, U^2 \left[1 + \frac{2}{3} \left(\frac{\pi r}{a} \right)^2 + \; . \; . \; . \; . \; . \right] \qquad [46q]$$

(3) *Flow Through a Grating.* When the stream is present, a similar surface S surrounds each of the points (0, 0), (0, $\pm a$), (0, $\pm 2a$) . . . . On the surface surrounding (0,a), for example, $\psi = aU$, and the equation of the surface can be written

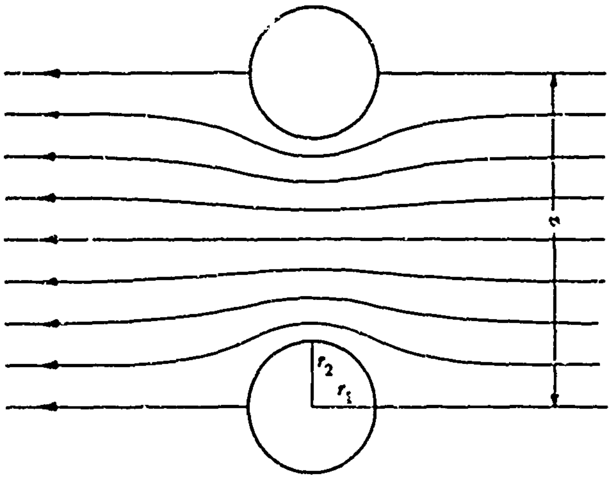


Figure 73 – Streamlines between two nearly circular cylinders, mounted in walls or forming two members of an infinite grating. See Section 46.

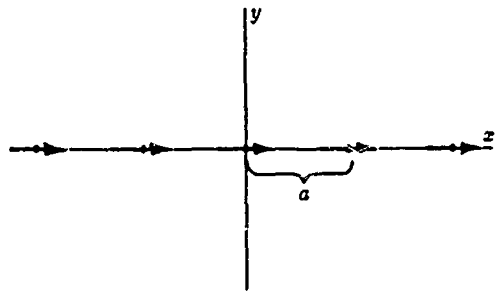


Figure 74 – Row of equal line dipoles with longitudinal axes.

$$U(y-a) = \frac{B}{H} \sin \frac{2\pi(y-a)}{2} \qquad [46r]$$

Thus, if no walls are inserted, the formulas represent a stream flowing at normal incidence through a grating whose bars are represented by the surfaces S. The bars are nearly circular in section provided their diameters are smaller than the intervening spaces. Some streamlines for this case are shown in Figure 73.

(For notation and method; see Section 34; Reference 1, Article 64.)

47. ROW OF EQUAL LINE DIPOLES ON A LONGITUDINAL AXIS; FLOW PAST A GRATING

$$w = B \cot \frac{\pi z}{a}, \quad a \text{ and } B \text{ real} \qquad [47a]$$

$$\phi = \frac{B}{H} \sin \frac{2\pi x}{a}, \quad \psi = -\frac{B}{H} \sinh \frac{2\pi y}{a} \qquad [47b,c]$$

$$H = 2\left[\sin^2 \frac{\pi x}{a} + \sinh^2 \frac{\pi y}{a}\right] = \cosh \frac{2\pi y}{a} - \cos \frac{2\pi x}{a} \qquad [47d]$$

$$u = \frac{2\pi B}{aH^2}\left(1 - \cos\frac{2\pi x}{a}\cosh\frac{2\pi y}{a}\right), \quad v = \frac{2\pi B}{aH^2}\,\sin\frac{2\pi x}{a}\,\sinh\frac{2\pi y}{a} \qquad [47e,f]$$

All functions are periodic in the direction of x with the period a, and the x-axis represents a plane of flow symmetry. The lines $x = 0, \pm a/2, \pm a, \ldots\ldots\ldots$ represent planes of purely geometrical symmetry.

Near the origin, as in Section 46, w reduces to $aB/\pi z$, so that there is a dipole at the origin with line-dipole moment equal to aB/π, and with its axis directed toward positive x if $B > 0$, toward negative x if $B < 0$. Similar line dipoles occur at $y = 0$ and $x = \pm a, \pm 2a, \ldots$. Hence the formulas represent the flow due a row of such dipoles spaced a apart along the x-axis; see Figure 74.

On the x-axis, $v = 0$, $u = \frac{2\pi B}{a}\left(1 - \cos\frac{2\pi x}{a}\right)^{-1}$ [47g]

On the lines $x = 0, \pm a, \pm 2a$, etc., $v = 0$,

$$u = -\frac{2\pi B}{a}\left(\cosh\frac{2\pi y}{a} - 1\right)^{-1} \qquad [47h]$$

In the *conjugate* flow the dipole axes are directed parallel to the y-axis; this is the flow of Section 46 rotated through 90 degrees.

Flow Past a Grating

If a uniform flow at velocity U toward negative x is superposed, a term $-U$ is added in u, Uz in w, and Ux in ϕ, and the formula for ψ becomes

$$\psi = Uy - \frac{B}{H}\sinh\frac{2\pi y}{a} \qquad [47i]$$

Assume that B/U is positive, so that the dipole axes are oppositely directed to the stream. Then $\psi = 0$ on the x-axis and on a dividing surface S, symmetrical with respect to both axes, given by

$$y = \frac{B}{U}\sinh\frac{2\pi y}{a} \Big/ \left(\cosh\frac{2\pi y}{a} - \cos\frac{2\pi x}{a}\right) \qquad [47j]$$

If $\pi B/aU > 1$, S consists of two undulating surfaces enclosing the plane $y = 0$. This appears from Equation [47k], which cannot be solved for r_1 if $\pi B/aU > 1$, so that S cannot cut the x-axis.

If $\pi B/aU < 1$, S breaks up into a set of similar cylindrical surfaces with axes at $x = 0, \pm a, \pm 2a \ldots$. The semidiameters of each cylinder in the x and y directions, found as in Section 46, are r_1 and r_2 where

$$\sin \frac{\pi r_1}{a} = \sqrt{\frac{\pi B}{aU}}\,, \quad r_2 = \frac{B}{U} \coth \frac{\pi r_2}{a} \qquad [47k,l]$$

Stagnation lines occur on the cylinders where $y = 0$.

The formulas will then represent streaming flow past a grating whose bars have the contours of the cylinders. If the diameter of the bars is smaller than the spacing between them, they are nearly circular in section, the y-diameter exceeding the x-diameter by less than 2 percent. Streamlines for such a case are shown in Figure 75; only half of the symmetrical diagram is illustrated. If the diameter is small, $r_1 = r_2 = (aB/\pi U)^{1/2}$, nearly.

(For notation and method, see Section 34.)

48. ALTERNATING LINE DIPOLES; DIPOLE MIDWAY BETWEEN WALLS, WITH PERPENDICULAR AXIS

$$w = i B/\left(\sin \frac{\pi z}{a}\right) \qquad [48a]$$

$$\phi = \frac{B}{G} \cosh \frac{\pi x}{a} \sin \frac{\pi y}{a}, \quad \psi = \frac{B}{G} \sinh \frac{\pi x}{a} \cos \frac{\pi y}{a} \qquad [48b,c]$$

$$G = \sinh^2 \frac{\pi x}{a} + \sin^2 \frac{\pi y}{a} = \frac{1}{2}\left(\cosh \frac{2\pi x}{a} - \cos \frac{2\pi y}{a}\right) \qquad [48d]$$

$$u = \frac{\pi B}{aG^2}\left(\cosh^2 \frac{\pi x}{a} + \cos^2 \frac{\pi y}{a}\right) \sinh \frac{\pi x}{a} \sin \frac{\pi y}{a} \qquad [48e]$$

$$v = -\frac{\pi B}{aG^2}\left(\sinh^2 \frac{\pi x}{a} - \sin^2 \frac{\pi y}{a}\right) \cosh \frac{\pi x}{a} \cos \frac{\pi y}{a} \qquad [48f]$$

Near (0,0), $w \to ia\, B/\pi z$, representing a line dipole of moment aB/π at the origin with its axis toward positive y (if $B > 0$). See Section 46. The entire field represents a row of such dipoles spaced a apart along the y-axis and with their axes directed alternately toward positive and toward negative y.

In the *conjugate flow* the dipole axes are parallel to the x-axis.

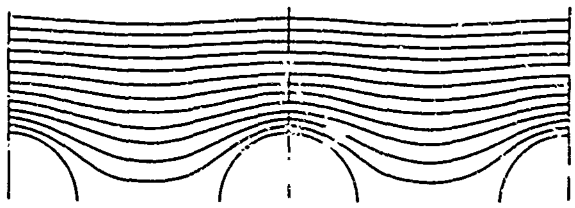


Figure 75 – Streamlines past a grating of nearly circular cylinders; see Section 47. (Copied from Reference 253.)

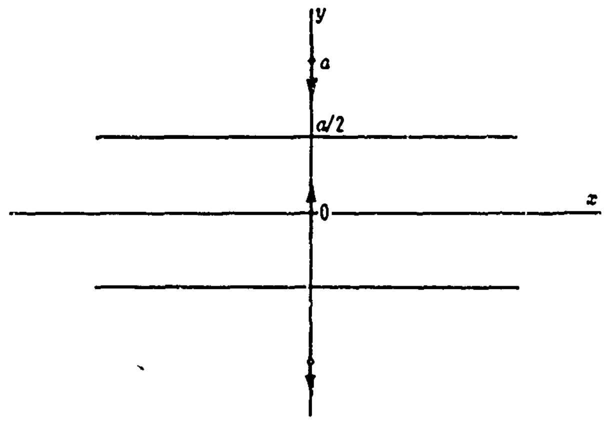


Figure 76 – Row of line dipoles with axes longitudinal but alternating in direction.

On the lines $y = \pm a/2$, $\psi = 0$. Hence parallel plane walls may be inserted along these lines. Then the flow is represented due to a dipole placed midway between walls a apart, with its axis perpendicular to the walls; see Figure 76.

$$\text{On the } x\text{-axis, } u = 0,\ v = -\frac{\pi B}{a} \coth\frac{\pi x}{a} \operatorname{csch}\frac{\pi x}{a} \qquad [48g]$$

$$\text{On the walls at } y = \pm a/2,\ u = \pm\frac{\pi B}{a} \tanh\frac{\pi x}{a} \operatorname{sech}\frac{\pi x}{a} \qquad [48h]$$

$$\text{On the } y\text{-axis, } v = \frac{\pi B}{a} \cot\frac{\pi x}{a} \csc\frac{\pi x}{a} \qquad [48i]$$

(For notation and method, see Section 34.)

49. LINE SOURCE, VORTEX OR DIPOLE ANYWHERE BETWEEN PARALLEL WALLS

If the line singularity between walls, as considered in Sections 44, 45, 46, and 48, is at a distance b from the median plane between the walls, the complex potential is modified as follows:

Line source:

$$w = -A\left[ln \sinh\frac{\pi(z-ib)}{2a} + ln \sinh\frac{\pi(z+ib-ia)}{2a}\right] \qquad [49a]$$

Line vortex:

$$w = \frac{i\Gamma}{2\pi}\left[ln \sinh\frac{\pi(z-ib)}{2a} - ln \sinh\frac{\pi(z+ib-ia)}{2a}\right] \qquad [49b]$$

Line dipole with axis parallel to walls:

$$w = \frac{C}{2}\left[\coth\frac{\pi(z-ib)}{2a} + \coth\frac{\pi(z+ib-ia)}{2a}\right] \qquad [49c]$$

Line dipole perpendicular to walls:

$$w = \frac{iC}{2}\left[\coth\frac{\pi(z-ib)}{2a} - \coth\frac{\pi(z+ib-ia)}{2a}\right] \qquad [49d]$$

The walls are assumed to be at $y = \pm a/2$ as before, where $a/2 > |b|$. If no walls are present, there are two rows of singularities displaced a distance $2b$ relatively to each other, the spacing in each row being $2a$. A, Γ, and C are real; the volume emitted per unit length from the source is $2\pi A$, the circulation around the vortex is Γ, the line-dipole moment is aC/π.

Expressions for ϕ, ψ, u, and v are easily constructed by substituting first $y - b$, then $y + b - a$, for y, and $2a$ for a, in formulas given in Sections 44, 45, 46, or 48, and combining the two terms thus obtained. For the line vortex the formulas for the conjugate flow of Section 44, not those of Section 45, are to be used, with $A = \Gamma/2\pi$; the corresponding complex potential is $w = iA \ln \sinh(\pi z/a)$. Similarly, for the fourth case, the conjugate flow of Section 46 reversed in sign is to be used, with a complex potential $iB \coth(\pi z/a)$; B is to be replaced by $C/2$. (Reference: Jaffe[30], Caldonazzo[32]).

50. TWO LINE DIPOLES IN OPPOSITION; DIPOLE AND A WALL

$$w = \mu\left(\frac{e^{i\alpha}}{z-c} - \frac{e^{-i\alpha}}{z+c}\right), \ \mu \text{ and } c \text{ real and } c > 0, \qquad [50a]$$

$$\phi = \mu\left(\frac{\cos(\theta_1-\alpha)}{r_1} - \frac{\cos(\theta_2+\alpha)}{r_2}\right), \ \psi = -\mu\left(\frac{\sin(\theta_1-\alpha)}{r_1} - \frac{\sin(\theta_2+\alpha)}{r_2}\right) \qquad [50b,c]$$

where the significance of r_1, r_2, θ_1, θ_2 is adequately shown in Figure 77.

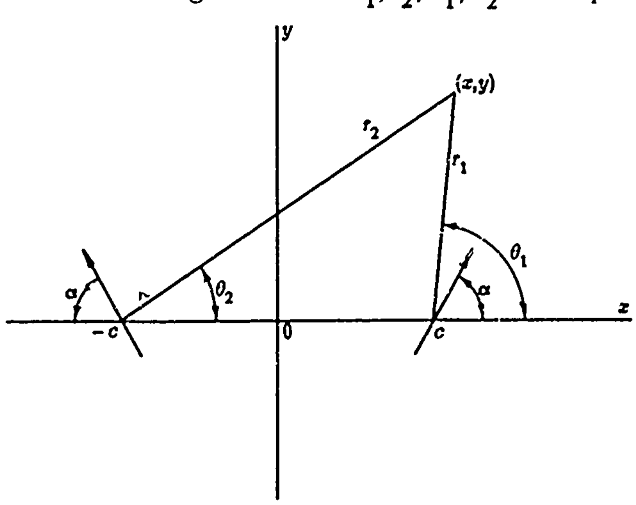


Figure 77 – A line dipole at $(c,0)$ and its image in a wall along the y-axis.

From Section 37 it is seen that these formulas represent a line dipole of moment μ, located at $(c,0)$ and having its axis directed at an angle α with the positive x-axis, together with another of equal moment located at $(-c,0)$ and having its axis inclined at a clockwise angle α from the negative x-axis. To obtain the second term as here written from [37r], replace α by $\pi-\alpha$ and note that $e^{i\pi} = -1$.

The particle velocity is easily found by adding vectorially the velocities due to the two dipoles.

Along the y-axis, where $\theta_2 = \pi - \theta_1$, and $r_1 = r_2$, $\psi = 0$. Hence a rigid wall may be inserted along the y-axis; then either half of the field represents the flow due to a line dipole in the presence of a parallel rigid plane boundary. The other dipole may be regarded as an image of the first in the wall.

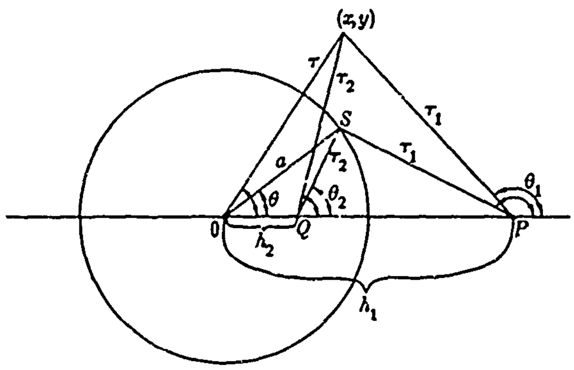
(For notation and method; see Section 34; Reference 2, Section 8.42.)

51. LINE SOURCE AND CYLINDRICAL BARRIER

The problem of a uniform line source parallel to a cylindrical barrier of circular cross-section is easily soluble by the method of images.

Let the source be at P, distant h_1 from the axis of the cylinder, whose radius is a; see Figure 78. Add, outside the cylinder, the flow that would be due to an equal and parallel line source on the inverse line Q in the cylinder, which lies in the plane containing the axis of the cylinder but at a distance $h_2 = a^2/h_1$ from the axis. Add also the flow due to a line sink of equal strength located on the axis 0 itself.

Figure 78 – A line source at P outside a circular cylinder.



The resultant stream function is from Equation [40c] in Section 40,

$$\psi(x,y) = -A(\theta_1 + \theta_2 - \theta) \qquad [51a]$$

where A is a real constant and θ_1, θ_2, θ are variable angles defined as shown in Figure 78. At any point S on the cylinder the triangles OSQ and OSP are similar; hence angle OSQ equals angle OPS or $\pi - \theta_1$, whence $\theta_2 = \theta + \pi - \theta_1$. Thus at points on the cylinder $\theta_1 + \theta_2 - \theta = \pi$ and $\psi = -A\pi$ and is constant. The streamline for $\psi = -A\pi$ thus proceeds from P to the nearest point on the cylinder, divides, passes around it, and then continues along the extension of PO. The extension of OP to the right of P is the streamline for $\psi = 0$.

The velocity potential is, from Equation [40b],

$$\phi = -A \log\left(\frac{r_1 r_2}{r}\right), \qquad [51b]$$

or, in a dimensionally balanced form,

$$\phi = -A \log\left(\frac{r_1 r_2}{ar}\right) \qquad [51c]$$

where r_1, r_2, and r are distances as shown in Figure 78. The complex potential, with the origin on the axis of the cylinder and the source at $(h_1, 0)$, is

$$w = -A\left[\log(z - h_1) + \log(z - h_2) - \log z\right] \qquad [51d]$$

The components of velocity may be written down from Equations [40e,f].

On the cylinder, taking $ds = a\,d\theta$,

$$q = -\frac{\partial\phi}{\partial s} = -\frac{1}{a}\frac{\partial\phi}{\partial\theta} = A\left(\frac{h_1}{r_1^2} + \frac{h_2}{r_2^2}\right)\sin\theta$$

since

$$r_1^2 = a^2 + h_1^2 - 2ah_1\cos\theta, \quad r_2^2 = a^2 + h_2^2 - 2ah_2\cos\theta,$$

or, using the similar triangles again to show that $r_2/r_1 = a/h_1$,

$$q = 2A\frac{h_1}{r_1^2}\sin\theta. \qquad [51e]$$

Half of the flow net, which is symmetrical with respect to the OP axis, is shown for $h_1 = 2a$ in Figure 79. The source is at P.

The total force per unit length on the cylinder is

$$F = \frac{2\pi\rho a^2 A^2}{h_1(h_1^2 - a^2)} \qquad [51f]$$

and is directed toward the source. This result is easily obtained from the Blasius theorem and the residues at $z = 0$ and $z = h_2$; compare Section 42.

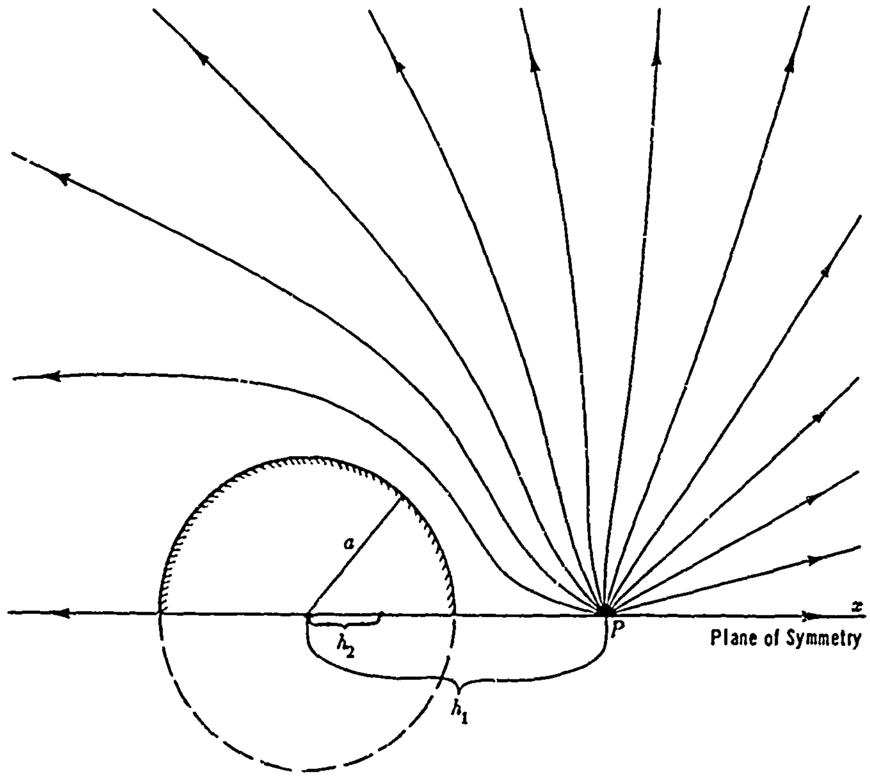


Figure 79 – Streamlines from a line source at P near a circular cylinder.

The equations will represent also the flow inside a cylindrical shell of radius a due to a line sink along the axis and a parallel line source of equal strength distant h_2 from the axis; then in the formulas the constant h_1 stands for a^2/h_2.

If A is made negative, sources become sinks and vice versa, and all velocities are reversed. (See Reference 1, Article 64; Reference 2, Sections 8.61, 8.62.)

52. LINE DIPOLE AND CYLINDRICAL BARRIER

Near a circular cylinder let there exist both a parallel line source and a line sink of equal strength. Then, upon superposing the flows as described in Section 51, it is noted that the images at the origin cancel each other and there remain only the image source and sink at the inverse points. By imagining the external source and sink to coalesce while suitably increasing in strength, so as to form a dipole, the conclusion is reached that the image of a line dipole in a circular cylinder with parallel axis is a dipole located on the inverse line with respect to the cylinder.

Let the given dipole be at a distance b_1 from the axis of the cylinder, whose radius is a, and let its axis make an angle α with the line drawn from the axis of the cylinder through the position of the dipole, as in Figure 80. Then, using [37r] for the potential due to a line dipole,

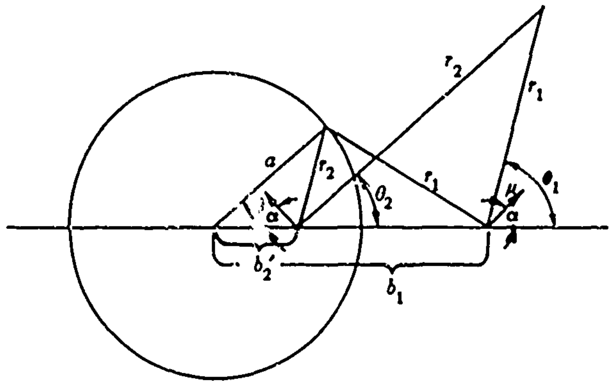


Figure 80 – Image of a line dipole in a circular cylinder; see Section 52.

$$w = \mu \left(\frac{e^{i\alpha}}{z-b_1} - \frac{a^2}{b_1^2}\frac{e^{-i\alpha}}{z-b_2}\right), \quad b_2 = \frac{a^2}{b_1}, \tag{52a,b}$$

$$\phi = \mu \left[\frac{\cos(\theta_1-\alpha)}{r_1} - \frac{a^2}{b_1^2}\frac{\cos(\theta_2+\alpha)}{r_2}\right], \quad \psi = -\mu\left[\frac{\sin(\theta_1-\alpha)}{r_1} - \frac{a^2}{b_1^2}\frac{\sin(\theta_2+\alpha)}{r_2}\right] \tag{52c,d}$$

where the significance of r_1, r_2, θ_1, and θ_2 is as shown in Figure 80. The real constant μ is the line-dipole moment of the given dipole. The dipole and its image have axes equally but oppositely inclined to the line joining their positions.

On the cylinder itself, since $r_2/r_1 = a/b_1$ and $r_1 \sin\theta_1 = r_2 \sin\theta_2$, expanding the sines

$$\psi = \mu \sin\alpha \left(\frac{\cos\theta_1}{r_1} + \frac{a^2}{b_1^2}\frac{\cos\theta_2}{r_2}\right)$$

But, also, on the cylinder

$$r_1 \cos\theta_1 = (a^2 - b_1^2 - r_1^2)/2b_1$$

and

$$r_2 \cos\theta_2 = \frac{1}{2b_2}\left(a^2 - b_2^2 - r_2^2\right) = \frac{1}{2b_1}\left(b_1^2 - a^2 - r_1^2\right)$$

Hence

$$\psi = -\frac{\mu \sin\alpha}{b_1} \tag{52e}$$

and thus has a constant value, so that the cylinder forms part of a stream surface.

The formulas may represent the flow due to a dipole outside of such a cylinder, or, if $b_1 < a$, the flow inside a cylindrical shell of radius a caused by a line dipole inside it. In either case the subscript 1 refers to the given dipole, and the other dipole may be regarded as the image of this one in the cylinder.

The force on the cylinder can be found, as in Section 42, from the Blasius theorem and the method of residues. Here

$$\frac{dw}{dz} = \mu\left[-\frac{e^{i\alpha}}{(z-b_1)^2} + \frac{a^2}{b_1^2}\frac{e^{-i\alpha}}{(z-b_2)^2}\right]$$

If $b_1 > a$, inside the cylinder there is only a singularity at $z = b_2$, and, expanding at $z = b_2$, as in Section 30,

$$\frac{1}{(z-b_1)^2} = (b_2-b_1+z-b_2)^{-2} = \frac{1}{(b_2-b_1)^2} - \frac{2(z-b_2)}{(b_2-b_1)^3} \cdots .$$

If $b_1 < a$, the singularity is located at b_1 and $(z-b_2)^{-2}$ is expanded; the path of integration is then traversed in the negative direction, so that $\oint (z-b_1)^{-1}dz = -2\pi i$. In either case the force per unit length on the cylinder is found to be directed toward the dipole and to be of magnitude

$$F_1 = \frac{4\pi\rho\mu^2 a^2 b_1}{(b_1^2-a^2)^3} \qquad [52h]$$

The force is thus independent of the direction of the dipole axis. (See Reference 2, Section 8.81, 8.82.)

53. LINE SOURCE IN UNIFORM STREAM

Upon the flow due to a line source at the origin let there be superposed a uniform flow at velocity U toward negative x. From Equations [35a] and [40a] the resultant w, ϕ, and ψ may be written

$$w = U(z - g\ln z) \qquad [53a]$$

$$\phi = U(x - g\ln r), \quad \psi = U(y - g\theta) \qquad [53b,c]$$

$$r = (x^2+y^2)^{1/2}, \quad \theta = \tan^{-1}(y/x),$$

where g is a real positive constant, $|\theta| \leqq \pi$, and θ has the sign of y; see Figure 81. Thus

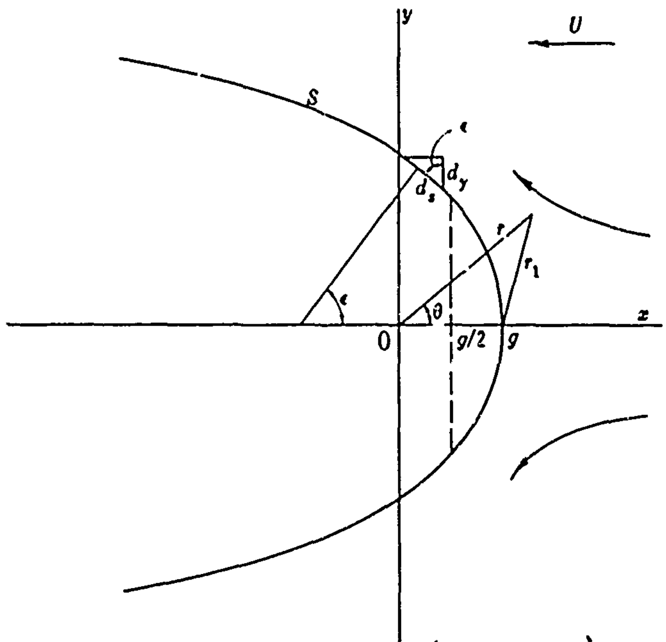


Figure 81 – Line source at 0 in a uniform stream; see Section 53.

$$u = U\left(-1 + g\,\frac{\cos\theta}{r}\right), \qquad v = gU\,\frac{\sin\theta}{r}, \qquad [53d,e]$$

$$\frac{dw}{dz} = U\left(1 - \frac{g}{z}\right) = U\,\frac{x + iy - g}{x + iy}, \qquad q = \left|\frac{dw}{dz}\right| = |U|\frac{r_1}{r}, \qquad r_1^2 = (x-g)^2 + y^2, \qquad [53f,g]$$

where r_1 is the distance from the stagnation point at $(g,0)$ Note that $\partial r/\partial x = x/r = \cos\theta$. The only singularity occurs at the origin; and the x-axis is an axis of flow symmetry.

The value $\psi = 0$ occurs on the positive x-axis and also on a curve S defined by

$$y = g\theta, \quad \text{or} \quad r = g\theta/\sin\theta, \qquad [53h,i]$$

As $\theta \to 0$, $y \to 0$; also $\theta/\sin\theta \to 1$, so that $r \to g$. Hence S passes through $(g,0)$. Toward $x = -\infty$, $|\theta| \to \pi$, and $|y|$ increases to a maximum of πg.

Thus the streamline for $\psi = 0$ follows the x-axis to the nose of S, where it divides and continues along both sides of S. Every other streamline undergoes a lateral displacement of $\pm\,\pi g$ from $+\infty$ to $-\infty$, or from $\theta = 0$ to $\theta = \pm\,\pi$. All fluid coming from infinity remains outside of S, and all fluid emitted from the source remains inside S.

An infinite solid cylinder may be inserted along S, extending to infinity also toward negative x, where it has a maximum thickness of $2\pi g$. The formulas then represent flow past this cylinder. They may also be used to represent the flow due to a line source inside a cylindrical shell having the form of S.

If the motion is steady, the values $q = |U|$ and $p = p_\infty$ occur on S where $r_1 = r$, $x = g/2$, from [53g]. On the x-axis ahead of S, $r = x$, $r_1 = x - g$, $q = |U|\,(1 - g/x)$, and

$$p - p_\infty = \tfrac{1}{2}\rho\,U^2\left(\frac{2g}{x} - \frac{g^2}{x^2}\right) \qquad [53j]$$

The net force on the cylinder is parallel to the x-axis, by symmetry; taken positive toward negative x, it is, per unit of length perpendicular to the flow,

$$F = \int p \cos \epsilon \, ds$$

where p is the pressure on the cylinder at any point, ds is an element of distance along S, and ϵ is the angle between the normal to ds and the x-axis; see Figure 81. But, also, $ds \cos \epsilon = dy$ where dy is the element of y corresponding to ds. Hence

$$F = \int p \, dy \qquad [53k]$$

Inserting $p = -\rho\, q^2/2$, also $r_1^2 = r^2 + g^2 - 2gr \cos \theta$, and using [53h,i], the contribution of the Bernoulli term to the pressure is found to be

$$F_B = -\tfrac{1}{2}\rho g U^2 \int_{-\pi}^{\pi} \left(1 - \frac{2 \sin \theta \cos \theta}{\theta} + \frac{\sin^2 \theta}{\theta^2}\right) d\theta$$

But

$$\int_{-\pi}^{\pi} \frac{\sin^2 \theta}{\theta^2}\, d\theta = -\frac{\sin^2 \theta}{\theta}\Bigg|_{-\pi}^{\pi} + 2\int_{-\pi}^{\pi} \frac{\sin \theta \cos \theta}{\theta}\, d\theta = 2\int_{-\pi}^{\pi} \frac{\sin \theta \cos \theta}{\theta}\, d\theta$$

Hence

$$F_B = -\tfrac{1}{2}\rho g U^2 \int_{-\pi}^{\pi} d\theta = -\pi \rho g U^2$$

The same result is obtained with $p = -\rho\, U^2/2$. Hence the total force on S is the same as if the pressure in the fluid were uniform throughout.

Half of the symmetric diagram of streamlines outside of S is illustrated in Figure 82. The excess of pressure $p - p_\infty$ along the x-axis and then over S is shown on an arbitrary scale.

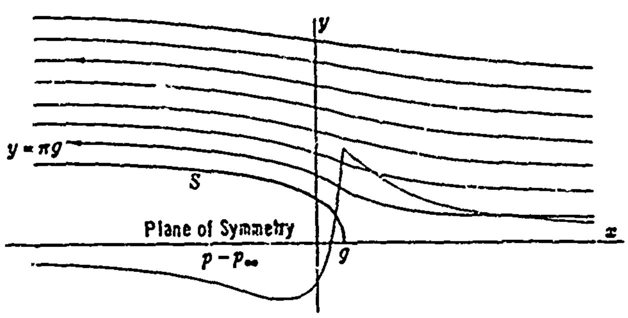


Figure 82 – Streamlines past a cylinder S of semi-infinite cross-section, only half of which is shown, of width $2\pi g$ at infinity; also, the distribution of pressure along the plane of symmetry and over the cylinder. Constructed with use of a line source at 0.

Changing the sign of U merely reverses all velocities, the source becoming a sink. S may be reversed in space by drawing the x-axis in the opposite direction. (For notation and method; see Section 34; Reference 2, Section 8.21.)

54. LINE SOURCE AND SINK IN UNIFORM STREAM.

Let a line source be located at $(a,0)$ and an equal sink at $(-a,0)$, and superpose uniform flow at velocity U toward negative x; see Figure 83. From Equations [35a] and [40a] the resultant potential and stream functions can be written, in terms of a real positive constant g,

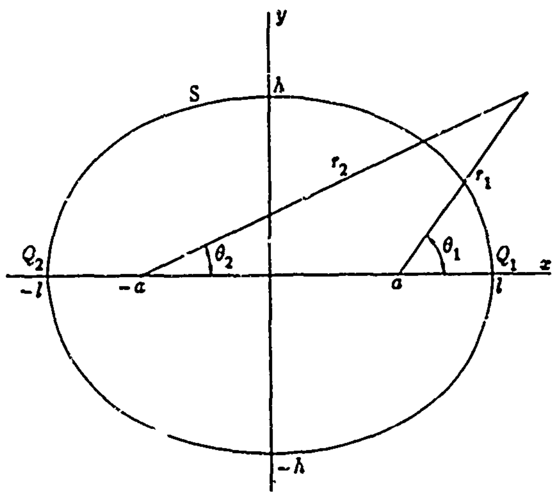


Figure 83 – Cylinder whose cross-section S is a Rankine oval, obtained from a line source at $(a,0)$ and a sink at $(-a,0)$. See Section 54.

$$w = U\{z + g\,[\ln(z + a) - \ln(z - a)]\} \qquad [54a]$$

$$\phi = U\left(x + g\ln\frac{r_2}{r_1}\right) = U\left(x + \frac{g}{2}\ln\frac{(x + a)^2 + y^2}{(x - a)^2 + y^2}\right) \qquad [54b]$$

$$\psi = U[y - g\,(\theta_1 - \theta_2)] = U\left(y - g'\tan^{-1}\frac{2ay}{x^2 + y^2 - a^2}\right) \qquad [54c]$$

where

$$r_1^2 = (x - a)^2 + y^2, \qquad r_2^2 = (x + a)^2 + y^2 \qquad [54c]$$

$$\tan\theta_1 = y/(x - a), \qquad \tan\theta_2 = y/(x + a)$$

and all angles may be supposed to lie between $-\pi$ and π and to have the sign of y. Also

$$u = U\left[-1 + g\left(\frac{x - a}{r_1^2} - \frac{x + a}{r_2^2}\right)\right] \qquad [54d]$$

$$v = g\,U\,y\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right), \qquad [54e]$$

$$q^2 = U^2\left[1 + \frac{4ag}{r_1^2\,r_2^2}\,(a^2 + ag - x^2 + y^2)\right] \qquad [54f]$$

Singularities occur at $z = \pm a$; and stagnation points Q_1, Q_2 occur where $dw/dz = 0$ or $z = x = \pm l$ and

$$l = \sqrt{a^2 + 2\,ag} \qquad [54g]$$

On the x-axis where $x > a$, $r_1 = x - a$, $r_2 = x + a$, and $q = |u|$ where

$$u = U\left(-1 + \frac{2ag}{x^2 - a^2}\right) \qquad [54h]$$

Thus the streamline for $\psi = 0$ follows the x – axis from $+\infty$ to Q_1, where it is joined by another branch coming from the source; then it divides and proceeds along the two halves of the curve S that is defined by

$$\frac{2ay}{x^2 + y^2 - a^2} = \tan\frac{y}{g} \qquad [54i]$$

From Q_2, one branch of this streamline proceeds to the sink, the other follows the x – axis to $-\infty$. The surface S divides the fluid into that which is coming from infinity and that which is on its way from the source to the sink.

The curve S is symmetrical about both axes, and is called a Rankine oval. Since an angle in radians and its tangent are nearly equal when the angle is small, the symbol *tan* can be omitted in [54i] when y is small; then, after canceling y, it becomes clear that the curve crosses the x – axis at the stagnation points. It is broadest in the middle, where $x = 0$. Its half-width h can be found by putting $y = k$ and $x = 0$ in [54i] and solving the resulting quadratic for h; the result can be written

$$\frac{a}{h} = \tan\frac{h}{2g} \qquad [54j]$$

On the middle circumference of S, $q = |u|$ and

$$u = -\,U\left(1 + \frac{2ag}{a^2 + h^2}\right) \qquad [54k]$$

Equations [54g] and [54j] can be written

$$\frac{l}{a} = \sqrt{1 + 2\frac{g}{a}}, \qquad \frac{a}{h} = \tan\left(\frac{h}{a}\,\frac{a}{2g}\right),$$

which shows that the shape of the oval as fixed by h/a and l/a depends only on the ratio g/a. As g/a increases, the oval comes to resemble an ellipse and finally approximates a circle; but for small g/a it is much more flat-sided and resembles the profile of a ship having a rounded bow and stern.

The formulas may represent the flow past a cylinder whose cross-section has the shape of the oval. In Figure 83 the oval is drawn for $g = a$. An example of the streamlines for $g/a = 0.17$ is shown in Figure 84. Here $l/a = 1.15$, $h/a = 0.41$, $h/l = 0.35$. Details of the construction according to the Maxwell-Rankine method are shown as described in Section 13; the parallel lines represent streamlines for the uniform flow, whereas the circular arcs diverging from a represent those for the flow due to the source and sink, all drawn for equal increments of ψ. In the original figure, however, twice as many lines and curves were drawn, for greater accuracy. The heavy curve is the cylinder S. Only one quarter of the diagram is shown, since it is symmetric with respect to both the x- and y-axes. According to the Bernoulli principle, the pressure excess, $p - p_\infty$, sinks from $\rho U^2/2$ at $(l,0)$, to zero at about the point indicated as P, and then remains negative to the middle, where, from [54f] with $x = 0$, $y = h$, $q = |u| = 1.29\,|U|$, $p - p_\infty = \rho(U^2 - q^2)/2 = -0.33\rho U^2$.

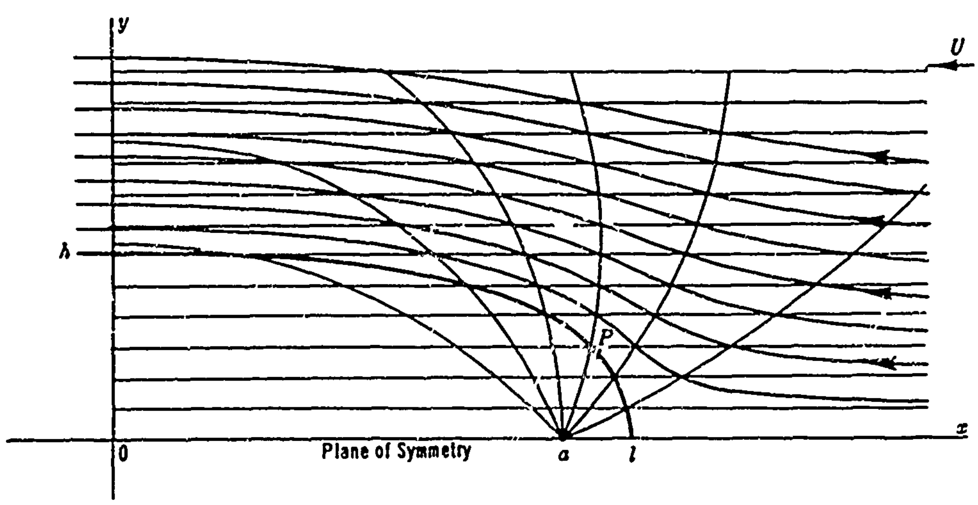


Figure 84 – One quarter of the streamline plot for a more slender Rankine oval. Construction of the plot by the Maxwell method is shown. See Section 54. (Copied from Reference 254.)

Another plot, also containing some of the construction lines and arcs, is shown in Figure 85. Here $g/a = 0.27$, $l/a = 1.24$, $h/a = 0.57$, $h/l = 0.46$.

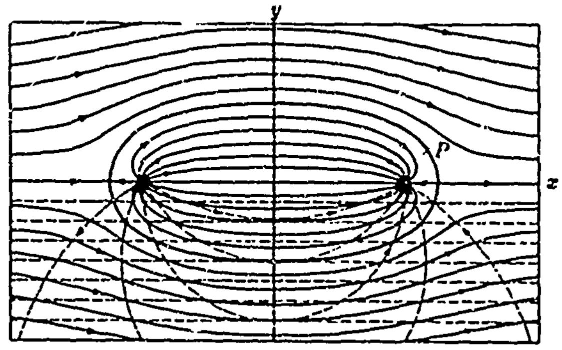


Figure 85 – Streamlines past a broader Rankine oval with construction curves. At P the pressure equals that at infinity. See Section 54. (Copied from Reference 7.)

Useful forms resembling the outlines of ships can be obtained in this manner; see McEntee[34], and Taylor[35].

The formulas may also represent the flow due to a line source and an equal line sink inside a cylindrical shell having the form of S.

Changing the sign of U merely interchanges source and sink and reverses all velocities.

Kinetic Energy

If the cylinder S is moving through fluid at rest at infinity, the term Uz is missing from w. Then, at large z,

$$w = gU \ln\left[\left(1+\frac{a}{z}\right)\Big/\left(1-\frac{a}{z}\right)\right] = gU \ln\left(1+\frac{2a}{z}\ldots.\right) = \frac{2agU}{z}+\ldots.$$

Hence, in Equation [76c] of Section 76, $b_1 = 2agU$, and, from [76d,f], the energy of the fluid, for unit thickness perpendicular to the flow, is

$$T_1 = \tfrac{1}{2}\rho(4\pi ag - S)\,U^2 \qquad [54k]$$

where S is the cross-sectional area of the cylinder.

(For notation and general explanation; see Section 34; Reference 2, Section 8.30.)

55. VORTEX PAIR IN A UNIFORM STREAM

The complex potential for a pair of line vortices with equal and opposite circulations, located at $(0,\pm c)$ and superposed upon a uniform flow at velocity U toward negative x, is

$$w = iA[\ln(z-ic) - \ln(z+ic)] + Uz \qquad [55a]$$

where A is a real constant. The circulation about $(0,c)$ is $\Gamma = 2\pi A$, that about $(0,-c)$, $\Gamma' = -2\pi A$. See Equations [40l] and [35a]. Hence

$$\phi = -A(\theta_1-\theta_2) + Ux, \qquad \psi = A\ln\frac{r_1}{r_2} + Uy \qquad [55b,c]$$

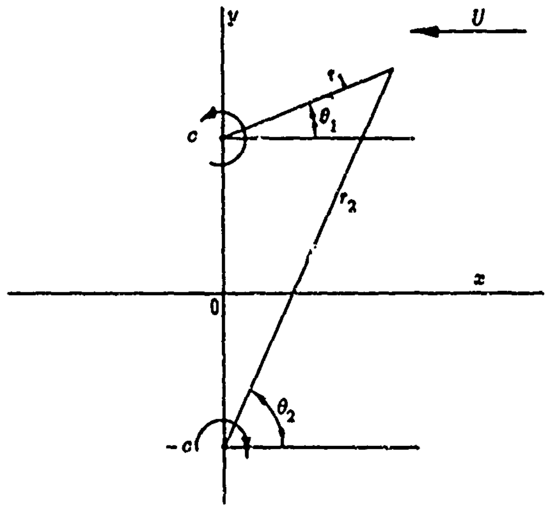


Figure 86 – Vortex pair in a transverse stream.

$$r_1 = \left[x^2+(y-c)^2\right]^{1/2}, \quad r_2 = \left[x^2+(y+c)^2\right]^{1/2}, \quad \theta_1 = \tan^{-1}\frac{y-c}{x}, \quad \theta_2 = \tan^{-1}\frac{y+c}{x}$$

See Figure 86. Also

$$u = -A\left(\frac{\sin\theta_1}{r_1} - \frac{\sin\theta_2}{r_2}\right) - U = 2Ac\,\frac{x^2+c^2-y^2}{r_1^2\,r_2^2} - U, \qquad [55d]$$

$$v = A\left(\frac{\cos\theta_1}{r_1} - \frac{\cos\theta_2}{r_2}\right) = \frac{4Acxy}{r_1^2\,r_2^2} \qquad [55e]$$

On the x-axis, $u = -U + 2Ac/(x^2+c^2)$; on the y-axis, $u = -U + 2Ac/(c^2-y^2)$. Hence, stagnation points occur; they are on the x-axis at $x = \pm x_Q$ if $A/U > c/2$, or on the y-axis at $y = \pm y_Q$ if $A/U < c/2$, where

$$x_Q = c\sqrt{\frac{2A}{cU} - 1}, \qquad y_Q = c\sqrt{1 - \frac{2A}{cU}} \qquad [55f,g]$$

If $A/U = c/2$, there is a single stagnation point at the origin.

The x-axis represents a plane of flow symmetry, the y-axis, a plane of geometrical symmetry for the flow net.

A dividing surface S always occurs, passing through the stagnation points. If $A/U \leqq c/2$, it consists of two loops, each surrounding one vortex. If $A/U > c/2$, it consists of a single loop surrounding both vortices, defined by the equation,

$$y = \frac{A}{U}\,ln\frac{r_2}{r_1}, \text{ or } x^2+y^2+c^2 = 2cy\coth(Uy/A). \qquad [55h,i]$$

In this case the streamline for $\psi = 0$ follows the x-axis and the curve S. That S passes through the stagnation points at $(\pm x_Q, 0)$ can be verified by first replacing coth (Uy/A) by A/Uy from the first terms of the hyperbolic series [33i].

The formulas may represent noncirculatory flow past a cylinder represented by the undivided curve S, or flow past two cylinders of a certain shape with circulation $\pm 2\pi A$ about them.

The most interesting case is that in which the vortices, when assumed to move with the fluid, actually stand still. This is realized when $A/2c = U$ or $A/U = 2c$, so that the velocity at either vortex due to the other just cancels the stream velocity U. Streamlines for this case are shown in Figure 86. The large oval curve is S; its semidiameters are 2.09c and 1.73c, approximately.

(For notation and method; see Section 34; Reference 1, Article 155; Reference 2, Section 13.30.)

56. OTHER COMBINATIONS INVOLVING LINE SOURCES OR DIPOLES

The following cases may be mentioned.

(1) Sources or dipoles only. Streamlines due to three equal and symmetrically placed line sources, with the fluid at rest at infinity, are shown in Figure 27 of Durand's Aerodynamic Theory[3], and for two sources and a sink in Figure 28. For a source and a dipole at the same point, see Reference 36.

(2) Source near a cylinder whose contour is elliptic or of certain other types: Morris[29] and Wrinch[37]; parabola-like: Sharpe[38]; a circular arc: Caldonazzo[39] and Sestini[40]; see Figure 87 for streamlines in one case with the source at the center of the circular arc.

(3) Source inside a rectangular cylinder: Jaffe[30] and Müller.[41]

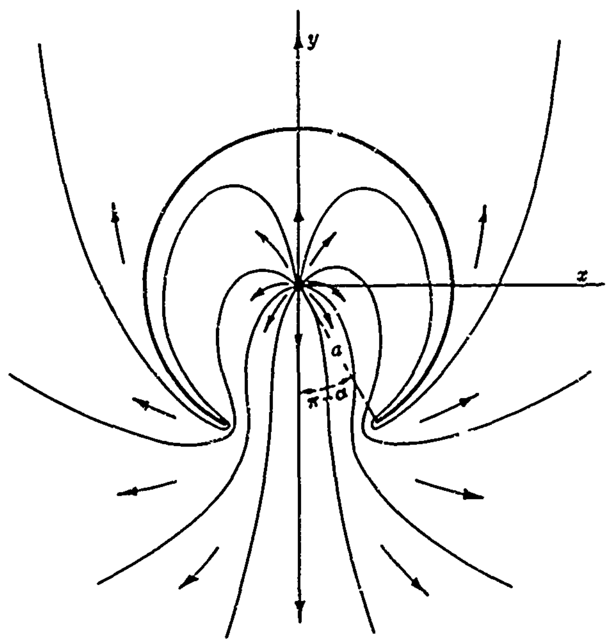


Figure 87 – Line source on axis of a circular-arc shell. (Copied from Reference 39.)

(4) Source or dipole on vertex of an angle (or sharply bent lamina): Agostinelli[42];

(5) Lamina, plane or bent along the median line, with a centered line source on one or both sides, and immersed in a stream: Colombo.[43] The velocity may be finite at the edges.

For a similar combination including circulation or a vortex, see Section 98.

(6) Sources or sinks in a stream. Boundaries of many other shapes then those described in Sections 53 and 54 can be obtained by inserting various combinations of sources and sinks into a uniform stream. There will always be a dividing surface along which a cylinder may be introduced. This surface is closed if the total strengths of sources and sinks are equal; otherwise it extends to infinity, in the direction of the stream if the sources are in excess, or in the opposite direction if sinks predominate. Although it is not always possible in this manner to match exactly the shape of an arbitrarily given cylinder, sufficiently close matches may often be secured. Uniform sheets of sources, as described in the next section, may be employed.

57. SHEETS OF LINE SOURCES OR VORTICES

Let sources be distributed uniformly over an infinite plane strip of width c. Let the source strength per unit area be α, so that a volume of fluid $2\pi\alpha$ is emitted per second from each unit area of the strip. Draw the xy-plane so that the given strip cuts it perpendicularly along the segment of the x-axis from $x = a$ to $x = b$; thus $c = b - a$. The resulting flow will then be parallel to the xy-plane.

On the strip, let x be replaced by x', and consider the sources on a substrip of width dx' extending from x' to $x' + dx'$; see Figure 88. Since the volume of fluid emitted by these sources is $2\pi\alpha dx$ per second, per unit of length perpendicular to the planes of flow, their contribution to the complex potential is, from [40a], in which now $A = \alpha dx$ and z is to be replaced by $z - x'$,

$$dw = - \alpha \ln (z - z') \, dx'$$

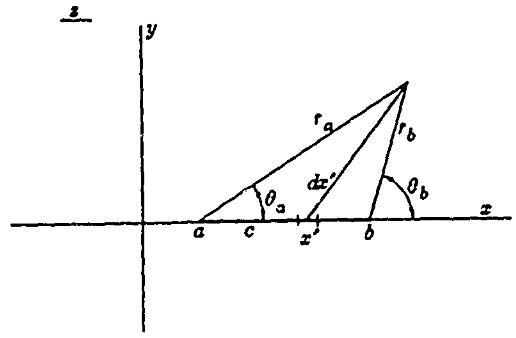


Figure 88 – A plane sheet of line sources.

where $z = x + iy$ and refers to a fixed point (x,y). Thus

$$w = -\alpha \int_a^b \ln(z - x')\, dx' = \alpha\left[(z - x') \ln(z - x') + x'\right] \Big/_a^b$$

or

$$w = \phi + i\psi = \alpha\left[(z - b) \ln(z - b) - (z - a) \ln(z - a)\right]$$

after dropping a constant term. Hence

$$\phi = \alpha\left[(x - b) \ln r_b - (x - a) \ln r_a - y(\theta_b - \theta_a)\right], \quad \psi = -\alpha\left[(x - a)\theta_a - (x - b)\theta_b + y \ln \frac{r_a}{r_b}\right] \qquad [57a,b]$$

where

$$r_a = \left[(x - a)^2 + y^2\right]^{1/2}, \quad r_b = \left[(x - b)^2 + y^2\right]^{1/2}, \quad \theta_b = \tan^{-1} \frac{y}{x - b}, \quad \theta_a = \tan^{-1} \frac{y}{x - a}$$

and θ_a and θ_b may be allowed to vary continuously without restriction. Also

$$u = -\frac{\partial \phi}{\partial x} = \alpha \ln \frac{r_a}{r_b}, \quad v = -\frac{\partial \phi}{\partial y} = \alpha(\theta_b - \theta_a) \qquad [57c,d]$$

The *conjugate flow* is that due to a uniform sheet of line vortices; the vortex strength or circulation per unit of width of the sheet is $2\pi\alpha$, and the circulation around the entire sheet is $2\pi\alpha c$; see Section 40.

If α is negative, the sources become sinks, or the direction of the circulation around the vortex sheet is reversed.

(For notation and method; see Section 34; Reference 3, p. 81.)

58. SOURCE SHEET IN A UNIFORM STREAM

Let the sheet of sources described in the preceding section be immersed in a uniform stream flowing at velocity U toward negative x. For simplicity let $a = -c$, $b = 0$, so that the sheet, of width c, extends from $x = -c$ to $x = 0$. Adding, from Section 35, a term Uz for the stream, and replacing α by gU, where g will be assumed to be positive:

$$w = U\{z + g\left[z \ln z - (z + c) \ln(z + c)\right]\} \qquad [58a]$$

$$\phi = U\{x + g\left[x \ln r - (x + c) \ln r_1 - y(\theta - \theta_1)\right]\} \qquad [58b]$$

$$\psi = U\left\{y + g\left[x\theta - (x + c)\,\theta_1 - y \ln \frac{r_1}{r}\right]\right\} \qquad [58c]$$

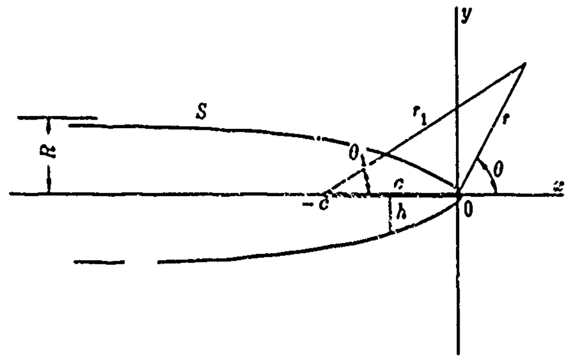


Figure 89 – Cylinder with semi-infinite cross-section S, obtained from a source-sheet in a stream. See Section 58.

$$u = U\left(-1 + g\ln\frac{r_1}{r}\right), \qquad v = g\,U\,(\theta - \theta_1), \qquad [58d,e]$$

where

$$r = (x^2 + y^2)^{1/2}, \quad r_1 = \left[(x+c)^2 + y^2\right]^{1/2}, \quad \theta = \tan^{-1}\frac{y}{x}, \quad \theta_1 = \tan^{-1}\frac{y}{x+c},$$

and θ and θ_1 may be assumed to lie between $-\pi$ and π and to have the sign of y; see Figure 89.

A stagnation point Q occurs on the x-axis, which represents a plane of symmetry, at $x = x_Q > 0$ where, to make $u = 0$, since $r = x$, $r_1 = x + c$,

$$x_Q = c(e^{1/g} - 1)^{-1} \qquad [58f]$$

On the positive x-axis, $\theta = \theta_1 = 0$ and $\psi = 0$, also, $\psi = 0$ on the dividing surface S defined by

$$y + g\left[x(\theta - \theta_1) - c\theta_1 - y\ln\frac{r_1}{r}\right] = 0 \qquad [58g]$$

By expanding all terms in powers of y it can be shown that S crosses the x-axis perpendicularly at Q. Since v has the same sign as y, the surface S, being a stream surface, must be broadest at $x \to -\infty$. At large distances from the origin, $\theta - \theta_1 \to c \sin\theta/r$ and $\ln(r_1/r) \to 0$, also, $x \to r\cos\theta$. Hence, Equation [58g] becomes, in the limit,

$$y + g(c\sin\theta\cos\theta - c\theta) = 0$$

Thus as $x \to -\infty$, y remains finite; hence $\theta \to \pi$ and $y \to cg\pi$. Hence, toward negative x, S becomes asymptotic to a cylinder of radius

$$R = \pi c g \qquad [58h]$$

The flow is thus represented past a semi-infinite cylinder whose profile in cross section is S. The cylinder has a sharper edge then that obtained with a line source in Section 53. Its shape is determined by g, its size by c, since increasing c and all coordinates in the same ratio leaves Equation [58g] satisfied. The half width at $x = -c/2$, or at the middle of the source sheet, where $r = r_1$ and $\theta = \pi - \theta_1$, is

$$y = h = \pi c g/2 = R/2 \qquad [58i]$$

An example is illustrated in Figure 90, for $g = 0.15$; $R \approx 0.47\, c$, $x = c/800$, approximately. The distribution of the excess of pressure above that at infinity, $p - p_\infty$ and of the velocity q, along the cylinder and along the plane of symmetry in front of it, are shown on arbitrary scales.

(For notation and method; see Section 34; Reference 3, p. 81.)

59. THE SIMPLER SINGULARITIES AND THEIR TRANSFORMATION

A simple type of singularity is the following. Suppose that at $z = c$ the complex potential w becomes infinite in such a way that near c it approximates the function $B \ln (z - c)$ where B is a constant; let the difference $w - B \ln (z - c)$ be a finite regular function of z even at $z = c$. Then, from the formulas in Section 40, it is clear that, if $B = A_1$ where A_1 is real, a line source exists at $z = c$, emitting $2\pi A_1$ units of volume of fluid per second and per unit length, whereas if $B = iA_2$ where A_2 is real, a line vortex exists there with circulation $2\pi A_2$ around it; if $B = A_1 + iA_2$, both source and vortex occur. Again, if w approximates similarly $\mu e^{i\alpha}/(z - c)$, where μ and α are real, then Equation [37r] shows that a line dipole exists at $z = c$, with line-dipole moment μ and with axis inclined at an angle α to the positive x-axis.

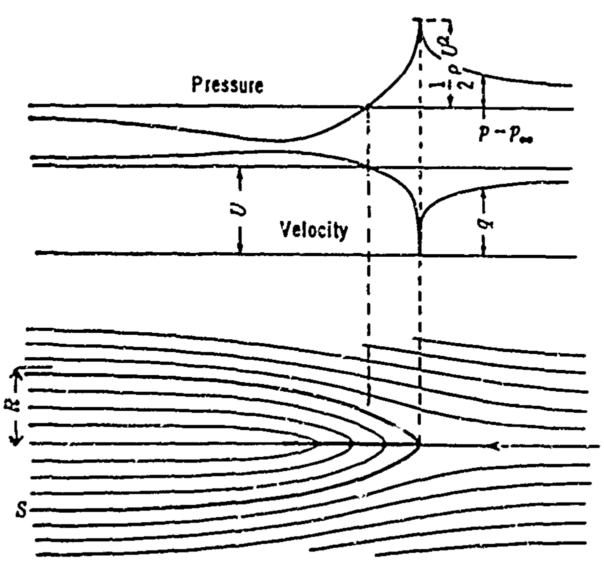


Figure 90 – Streamlines past the cylinder shown as S in Figure 89. The inner streamlines are those due to the source sheet alone inside a cylindrical shell of contour S. The pressure and the velocity are shown, along the axes and over the cylinder. See Section 58. (Copied from Reference 7.)

The flow due to the source, vortex or dipole may be regarded as superposed upon a flow that has no singularity at $z = c$.

If a transformation is now made to the plane of a new variable $\zeta = f(z)$, then any line source or vortex preserves its nature and strength on the ζ-plane, provided it occurs at a conformal point for the transformation at which and near which $d\zeta/dz$ exists and does not vanish. A dipole also transforms into a dipole, but, in general, with a different moment and axial direction.

For, if $\gamma = f(c)$, so that γ is the point on the ζ-plane corresponding to $z = c$,

$$B\ln(\zeta - \gamma) = B\ln\left[f(z) - f(c)\right] = B\ln(z - c) + B\ln\frac{f(z) - f(c)}{z - c}$$

The last term reduces to $B\ln\left[df/dz\right]$ at $z = c$ and hence represents a regular function at and near this point; and the coefficient of $\ln(\zeta - \gamma)$ is the same as that of $\ln(z - c)$. Thus a source and vortex are conserved in the transformation.

Similarly,

$$\frac{\mu' e^{i\alpha'}}{\zeta - \gamma} = \mu' e^{i\alpha'}\frac{z - c}{f(z) - f(c)}\,\frac{1}{z - c},$$

in which

$$\frac{f(z) - f(c)}{z - c} \to \frac{df}{dz} = \frac{d\zeta}{dz}$$

as $z \to c$;

or,

$$\frac{\mu' e^{i\alpha'}}{\zeta - \gamma} = \frac{\mu e^{i\alpha}}{z - c}$$

where $\mu' = r\mu$ and $\alpha' = \alpha + \theta$, r and σ being modulus and amplitude of $d\zeta/dz = re^{i\theta}$. Thus a dipole transforms into a dipole with its strength increased in the ratio of the modulus, and its axial inclination to the real axis increased by the amplitude, of the transformation.

(For notation; see Section. 34)

60. LINE SINGULARITY IN AN ANGLE

Consider the transformation

$$\zeta = z^n, \qquad z = \zeta^{1/n}, \qquad [60a,b]$$

where n is a real number not less than 1/2. In terms of $\zeta = re^{i\theta}$,

$$z = r^{1/n} e^{i\theta/n} \qquad [60c]$$

Thus the real axis of ζ, corresponding to $\theta = 0$ or π, s bent at the origin into two radii on the z-plane enclosing an angle π/n, and the space above the real axis of ζ is transformed into the space in this angle. The angle is concave if $n > 1$, convex if $1/2 \leqq n < 1$. Under the transformation, the upper half of the ζ-plane may be imagined to expand or contract as the negative half of the real axis rotates into the proper position.

An infinite wall lying along the real axis of ζ thus becomes an angle formed by two semi-infinite planes joined at their edges. Suppose that there is also a singularity in the flow on the ζ-plane, such as a source, vortex or dipole, located at a point which lies above the x-axis and is represented by $\zeta = h^n e^{i\beta n}$ or ($h^n \cos \beta n$, $h^n \sin \beta n$). Assume that $0 \leqq \beta \leqq \pi/n$. This singularity will have a line image in the plane wall, located as if behind a mirror or at $\zeta = h^n e^{-i\beta n}$. The effect of the transformation [60a,b] will then be to transform this singularity into a similar one on the z-plane, located at $z = h e^{i\beta}$ or ($h \cos \beta$, $h \sin \beta$) inside or on the angle.

Equations [43a], [42a], and [50a] are readily adapted to the geometry of the present case by changing z to ζ and making the proper substitution for c or h. Let all amplitudes θ, except that of $e^{-i\beta n}$, be taken in the range $0 \leq \theta < 2\pi$. The complex potentials w on the z-plane thus obtained are, when written in terms of z,

$$\text{Source:} \quad w = -A\left[\ln(z^n - h^n e^{i\beta n}) + \ln(z^n - h^n e^{-i\beta n})\right].$$

$$\text{Vortex:} \quad w = iA\left[\ln(z^n - h^n e^{i\beta n}) - \ln(z^n - h^n e^{i\beta n})\right].$$

$$\text{Dipole:} \quad w = \mu\left[e^{i\alpha}(z^n - h^n e^{i\beta n})^{-1} - e^{-i\alpha}(z^n - h^n e^{-i\beta n})^{-1}\right].$$

According to the results of the last section, the source on the z-plane emits a volume of fluid equal to $2\pi A$ per unit length, and the circulation around the vortex is still $\Gamma = 2\pi A$, as on the ζ-plane. The transformed dipole moment, however, is $\mu/(nh^{n-1})$, and its axis is directed at an angle $\alpha-(n-1)\beta$ to the positive x-axis on the z-plane. For, as $z \to he^{i\beta}$,

$$\frac{\mu e^{i\alpha}}{z^n - h^n e^{i\beta n}} = \frac{\mu e^{i\alpha}}{z - he^{i\beta}}\left(\frac{z^n - h^n e^{i\beta n}}{z - he^{i\beta}}\right)^{-1} \to \frac{\mu e^{i\alpha}}{z - he^{i\beta}} \Big/ \left(\frac{d}{dz} z^n\right)_{z = he^{i\beta}} = \frac{\mu}{nh^{n-1}} \frac{e^{i[\alpha-(n-1)\beta]}}{z - he^{i\beta}}$$

If $n = \frac{1}{2}$, the "angle" becomes a semi-infinite plane, as in Section 39. A few of the streamlines due to a symmetrically placed vortex near such a plane are illustrated in Figure 91. In this figure $\beta = \pi$, so that on the ζ-plane the vortex lies on the y-axis and its streamlines are circles, like those shown in Figure 61 and suggested briefly in Figure 92. References: Greenhill[26], Hamel[44], Paul[45,46], and fo the vortex, Miyadzu[47]; with flow past the corner and perhaps finite q at the edge, Uslenghi[48]; with a line source on the edge, Kucharski[49].

(For notation and method; see Section 34: Reference 2, Section 8.51.)

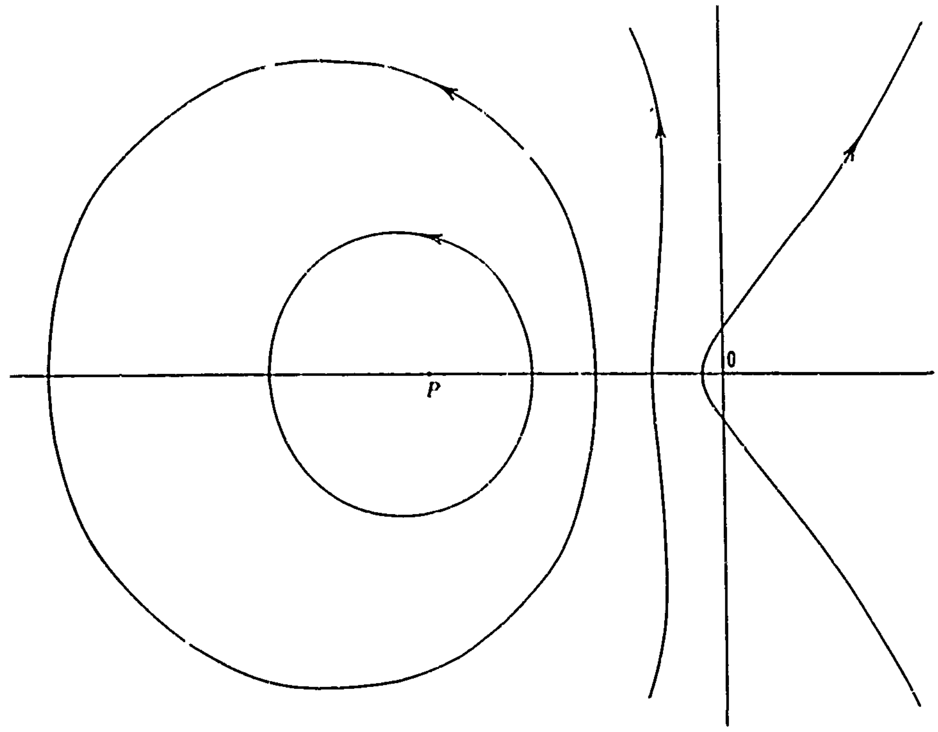


Figure 91 – Streamlines due to a line vortex at P, opposite a semi-infinite rigid plane extending from 0 toward the right.

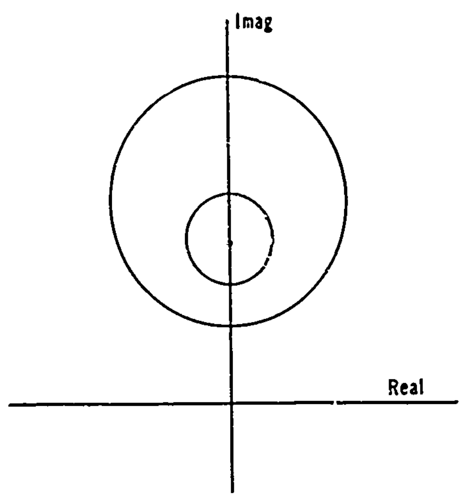


Figure 92 – The vortex of Figure 91 before transformation to the z-plane.

TRANSFORMATIONS DEFINED INVERSELY

61. ELLIPSES AND HYPERBOLAS

$$z = c \cosh w \qquad [61a]$$

Here c is a real positive constant. Since $z = x + iy$, $w = \phi + i\psi$,

$$x = c \cosh\phi \cos\psi, \qquad y = c \sinh\phi \sin\psi, \qquad [61b,c]$$

$$\frac{dw}{dz} = \left(\frac{dz}{dw}\right)^{-1} = \frac{1}{c \sinh w} = \frac{1}{c(\sinh\phi \cos\psi + i \cosh\phi \sin\psi)},$$

$$q^2 = \left|\frac{dw}{dz}\right|^2 = \frac{1}{c^2 G}, \quad G = \sinh^2\phi \cos^2\psi + \cosh^2\phi \sin^2\psi, \qquad [61d,e]$$

or

$$G = \sinh^2\phi + \sin^2\psi = \tfrac{1}{2}(\cosh 2\phi - \cos 2\psi), \qquad [61f]$$

by the use of hyperbolic formulas listed in Section 32.

By substitution it can readily be verified that the solutions of [61b,c] for ϕ and ψ can be written

$$\cosh\phi = \frac{1}{2c}\left\{\left[(x+c)^2 + y^2\right]^{1/2} + \left[(x-c)^2 + y^2\right]^{1/2}\right\}, \qquad [61g]$$

$$\cos\psi = \frac{1}{2c}\left\{\left[(x+c)^2 + y^2\right]^{1/2} - \left[(x-c)^2 + y^2\right]^{1/2}\right\} \qquad [61h]$$

Here the positive square root is meant. The sign of ϕ and the value of ψ must be chosen to fit [61b,c].

Singular points occur wherever both $\phi = 0$ and $\sin\psi = 0$, so that $dw/dz \to \infty$, hence at (c, o) and $(-c, o)$. Furthermore, two types of multiplicity occur: ψ is many-valued with a period of 2π; and the same point on the z-plane corresponds to $-\phi_1, -\psi_1$, as to ϕ_1, ψ_1. The latter multiplicity extends to dw/dz, which has opposite signs for $-\phi_1, -\psi_1$, and for ϕ_1, ψ_1.

By elimination of ϕ or ψ it is found that

$$\frac{x^2}{c^2\cosh^2\phi} + \frac{y^2}{c^2\sinh^2\phi} = 1, \qquad \frac{x^2}{c^2\cos^2\psi} - \frac{y^2}{c^2\sin^2\psi} = 1. \qquad [61i,j]$$

Thus the curves ϕ = constant are ellipses, while the curves ψ = constant are hyperbolas; both families of curves are confocal, with common foci at $(\pm c, 0)$, and, as usual, they cut each other orthogonally. They are illustrated in Figure 93, also, in more detail, in Figure 129, on which ξ may be identified with ϕ and η with ψ. Two of the hyperbolas degenerate into parts of the x-axis:

$$\psi = 0, \ x = c\cosh\phi > c; \qquad \psi = \pi, \ x = -c\cosh\phi < -c,$$

using [61b,c]. Again, on the y-axis, $\psi = \pi/2$ or $3\pi/2$ and $y = \pm c \sinh\phi$. The ellipse for $\phi = 0$ degenerates into the x-axis between $\pm c$, on which $x = c\cos\psi$.

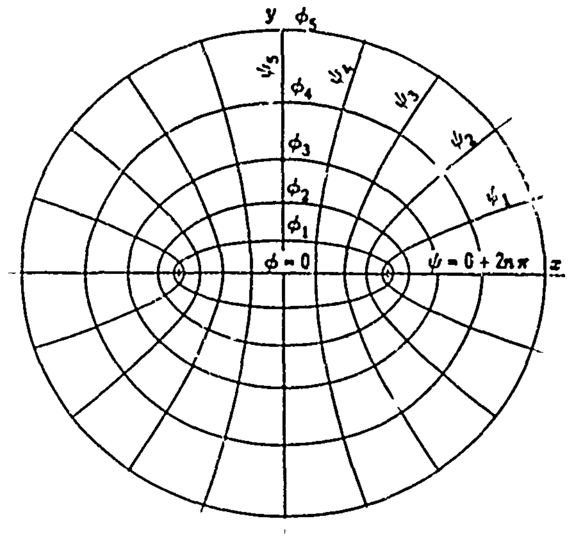


Figure 93 – Confocal ellipses and hyperbolas; the flow now on the z-plane for $z = c \cosh(\phi + i\psi)$.

In the hydrodynamical applications, the double-valued character of dw/dz makes it necessary to insert boundaries so as not only to exclude the singular points but also to prevent the fluid from circulating around just one of them. When this has been done, a choice can be made for the values of ϕ and ψ such that they vary continuously throughout the fluid and such that their derivatives represent a single-valued velocity. The singular points cannot be interpreted as representing a source and a sink, either simple or compound; mathematically, they are not poles but branch points.

Flow between Hyperbolic Cylinders

If ϕ is taken as the potential, boundaries may be inserted along any two of the ψ hyperbolas. The formulas then represent the flow between two hyperbolic cylinders. Convenient ranges for the variables are:

$$-\infty < \phi < \infty, \qquad 0 \leq \psi \leq \pi.$$

Since $dw/dz = -u + iv$, from [61a–f]

$$u = -\frac{1}{cG}(\sinh\phi \cos\psi), \qquad v = -\frac{1}{cG}\cosh\phi \sin\psi. \qquad [61k,l]$$

On a boundary defined by $\psi = \psi_1$, from [61c,d] and [61f],

$$q = \frac{1}{c}\left(\sin^2\psi_1 + \frac{y^2}{c^2 \sin^2\psi_1}\right)^{-1/2} \qquad [61m]$$

On the y-axis, $\psi = \pi/2$ and $q = 1/\sqrt{c^2 + y^2}$. In the plane of the opening $\phi = 0$ and the velocity is from Equations [61d,f,b],

$$q = \frac{1}{c \sin \psi} = \frac{1}{\sqrt{c^2 - x^2}} \qquad [61n]$$

The volume of fluid that flows per second through unit length of the slot is represented by the increase in ψ from one hyperbolic face to the other.

Flow through a slot

If the cylinders are allowed to shrink onto the x-axis, the flow becomes that through a slot of width $2c$ in a plane solid sheet extending to infinity on both sides, as illustrated in Figure 94a. The volume passing per second, per unit length of the slot, is then π. On either face of the solid sheet $\cos \psi = \pm 1$, $x = \pm c \cosh \phi$ and, from [61d,e]

$$q = \frac{1}{c\,|\sinh \phi|} = \frac{1}{\sqrt{x^2 - c^2}} \qquad [61o]$$

Thus $q \to \infty$ at the edges of the slot.

If, for generality, ϕ and ψ are replaced in all formulas by ϕ/k and ψ/k, all velocities and the issuing volume are multiplied by k.

Circulatory Flow Around an Elliptic Cylinder or Plane Lamina

For the *conjugate flow*, with potential $\phi' = \psi$ and stream function $\psi' = -\phi$, an elliptical cylinder may be inserted along one of the ψ' or ϕ curves, or a plane lamina of width $2c$ along the x-axis between $\pm c$. The irrotational flow around such a cylinder or lamina is then represented. This is illustrated in Figure 94b, where any one of the curves may represent the

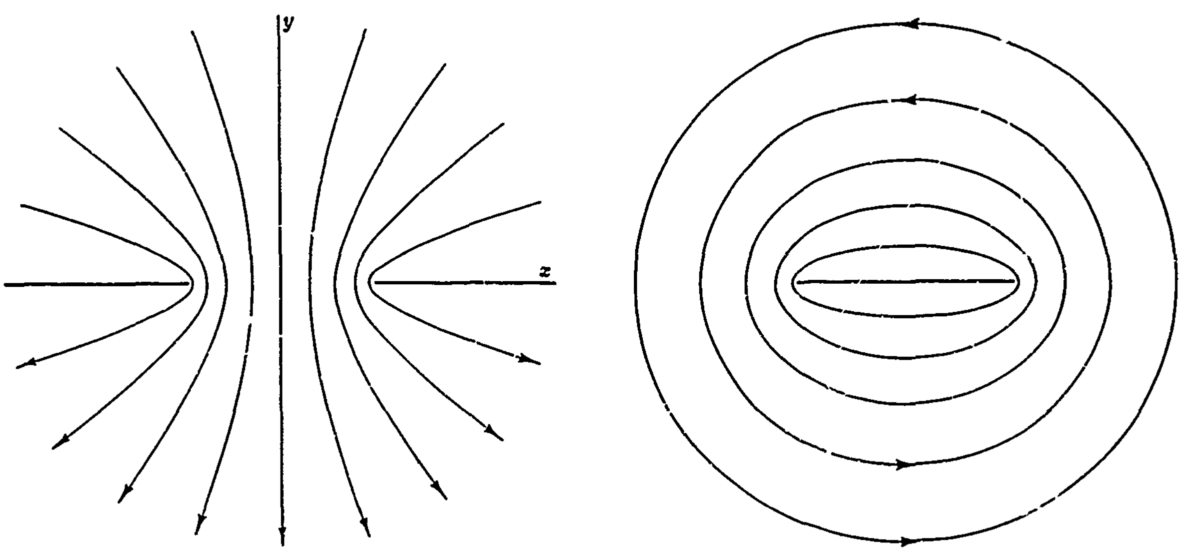


Figure 94a Figure 94b

Figure 94 – Streamlines for (a) flow through a slot in an infinite plane wall, (b) circulatory flow around a plane lamina.

cylinder, or the line the lamina, the outer curves then representing streamlines. For this case it is most convenient to keep $\phi > 0$, hence $\psi' < 0$. Then from [61b,c] it is easily verified that, as ϕ' or ψ increases continuously from 0 to 2π, the point (x,y) passes once around the cylinder. Thus the many-valuedness of ϕ' implies the existence of circulation around the cylinder of magnitude 2π.

On an elliptical cylinder defined by $\psi' = -\phi = -\phi_1$, with major semiaxis $a_1 = c \cosh \phi_1$, from [61d,f,b], after inserting $\sin^2 \psi = 1 - \cos^2 \psi$,

$$q = \frac{1}{c}\left(\cosh^2 \phi_1 - \frac{x^2}{c^2 \cosh^2 \phi_1}\right)^{-1/2} = \frac{1}{c}\left(\frac{a_1^2}{c^2} - \frac{x^2}{a_1^2}\right)^{-1/2} \qquad [61p]$$

At large distances the elliptical streamlines approximate circles, and, since $\sinh \phi$ becomes large and nearly equal to $\cosh \phi$, it is readily seen with the use of [61e] that

$$q = \frac{1}{(x^2 + y^2)^{1/2}}$$

approximately. Thus the flow approximates that of a line vortex (Section 40) at the origin.

If ϕ' and ψ' are replaced in these last formulas by ϕ'/k, ψ'/k, then all velocities are multiplied by k and the magnitude of the circulation becomes $2\pi k$.

The variables ϕ and ψ, defined in terms of x and y by [61b,c], can be used as coordinates on the xy-plane; this use, and the geometrical properties of the transformation, are discussed in Section 82.

Among other possible forms, $z = ic \cosh w$ gives the same field of flow rotated through 90 deg, with the foci at $(0, \pm c)$; $z = c \cos w$ or $z = c \cosh (iw)$ is the conjugate transformation, in which ϕ, ψ are replaced by ψ, $-\phi$; and $z = c \sinh w$ gives the original field rotated through 90 deg and with ψ increased by $\pi/2$, to which the conjugate transformation is $z = ic \sin w$.

(For notation and method; see Section 34; Reference 1, Article 66; Reference 2, Sections 6.10, 6.30.)

62. STRAIGHT SPOUT

$$z = w + e^w. \qquad [62a]$$

Since

$$e^w = e^{\phi + i\psi} = e^\phi (\cos \psi + i \sin \psi), \quad z = x + iy, \text{ and } -u + iv = dw/dz,$$

$$x = \phi + e^\phi \cos \psi, \qquad y = \psi + e^\phi \sin \psi, \qquad [62b,c]$$

$$\frac{dw}{dz} = \left(\frac{dz}{dw}\right)^{-1} = \frac{1}{1 + e^{w}} = \frac{1}{1 + e^{\phi}\,(\cos\psi + i\sin\psi)}$$

$$q = \left|\frac{dw}{dz}\right| = \frac{1}{G}, \quad G = (1 + e^{2\phi} + 2\,e^{\phi}\cos\psi)^{1/2} \qquad [62d,e]$$

$$u = -\frac{1}{G^2}(1 + e^{\phi}\cos\psi), \qquad v = -\frac{1}{G^2}\,e^{\phi}\sin\psi. \qquad [62f,g]$$

Since $1 - 2\,e^{\phi} + e^{2\phi} = (1 - e^{\phi})^2 \geqq 0$, it follows that $2\,e^{\phi} \leqq 1 + e^{2\phi}$; and the equality sign holds only if $\phi = 0$. Hence $G = 0$ and $dw/dz \to \infty$ only if $\phi = 0$ and $\cos\psi = -1$. Thus singular points occur at $x = -1$ and $y = \pm\,\pi,\ \pm\,3\pi,\ \pm\,5\pi\,.\,.\,.\,.\,.$

The streamline for $\psi = 0$ is the x-axis, on which $x = \phi + e^{\phi}$. Along this streamline, while ϕ is negative and numerically large, $\phi = x$, approximately. As ϕ increases, x increases, and at $\phi = 0$, $x = 1$; as $\phi \to +\infty$, $x \to +\infty$. Again, if $\psi = \pm\,\pi$, $y = \pm\,\pi$ and $x = \phi - e^{\phi}$. Here x increases to a maximum of -1 at $\phi = 0$, and then returns toward $-\infty$ as $\phi \to +\infty$. The two straight lines on which $y = \pm\,\pi$ and $x \leqq -1$ may be regarded as streamlines bent back on themselves. The intermediate streamlines, for $-\pi < \psi < \pi$, lie between these straight lines; for large negative ϕ they are almost parallel, but for large positive ϕ they fan out and cover the entire z-plane. Half of the flow net, which is symmetrical about the x-axis, is shown in Figure 95. For $|\psi| > \pi$, curves are obtained which overlap some of those already obtained; since this results in multiple-valued velocities such values of ψ cannot be used.

If plane, semi-infinite boundaries are inserted along the two straight streamlines, a motion is represented in which the fluid is flowing into and through a spout or mouthpiece bounded by two parallel walls $2\,\pi$ apart.

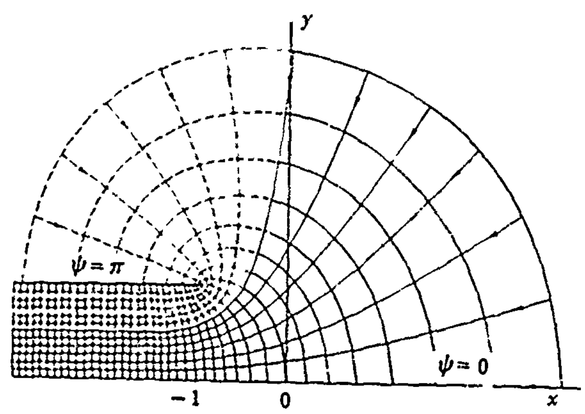


Figure 95 – Half of the symmetrical flow net for fluid entering a straight two-dimensional spout extending toward the left to infinity. (Copied from Reference 253.)

Within the spout, except near the entrance, ϕ is large and negative and $q = 1$, approximately. The total volume of the inflow per second, per unit of length perpendicular to the flow, is the value of $\Delta\psi$ between the walls or 2π. Along either wall $q = |u| = (1 - e^{\phi})^{-1}$; hence at the edge, where $\phi = 0$, $q \to \infty$. Inside the spout, $\phi \to -\infty$ as $x \to -\infty$ and $q \to 1$, whereas outside $\phi \to +\infty$ along any streamline and $q \to 0$. On the central or x-axis, $q = |u| = (1 + e^{\phi})^{-1}$.

The more general transformation

$$z = c(gw + e^{gw}) \qquad [62h]$$

or

$$x = c[g\phi + e^{g\phi}\cos(g\psi)], \quad y = c[g\psi + e^{g\phi}\sin(g\psi)],$$

where c and g are real constants and $c > 0$, represents a spout $2\pi c$ wide; for, when $g\psi = \pm\pi$, $y = \pm\pi c$. All velocities are changed from the values stated previously in the ratio $1/cg$. If $g < 0$ the fluid is issuing from the spout, but the flow pattern is the same. On the walls, $\psi = \pm\pi/g$, and the volume of outflow is $2\pi/g$.

(For notation and method; see Section 34; Reference 1, Article 66.)

63. DIVERGING SPOUT

$$z = \frac{1-n}{n}(1 - e^{-nw}) + e^{(1-n)w}, \quad 0 < n < \tfrac{1}{2}; \qquad [63a]$$

$$x = \frac{1-n}{n}[1 - e^{-n\phi}\cos(n\psi)] + e^{(1-n)\phi}\cos[(1-n)\psi], \qquad [63b]$$

$$y = \frac{1-n}{n}e^{-n\phi}\sin(n\psi) + e^{(1-n)\phi}\sin[(1-n)\psi], \qquad [63c]$$

$$\frac{dz}{dw} = (1-n)(1 + e^{w})e^{-nw}.$$

This is a generalization of the preceding transformation, to which it reverts if $n \to 0$.

The streamline for $\psi = 0$ is again the x-axis. If $\psi = \pm\pi$, since $\cos(1-n)\pi = -\cos n\pi$, $\sin(1-n)\pi = \sin n\pi$,

$$x = \frac{1-n}{n} - \left(\frac{1-n}{n} + e^{\phi}\right)e^{-n\phi}\cos n\pi,$$

$$y = \pm\left(\frac{1-n}{n} + e^{\phi}\right)e^{-n\phi}\sin n\pi$$

or, as may be verified by substituting the formula for x,

$$y = \pm \left(\frac{1-n}{n} - x\right) \tan n\pi.$$

Thus the streamlines for $\psi = \pm \pi$ are straight lines inclined at angles $\mp n\pi$ to the positive x-axis. These lines do not cross the axis but end at the points

$$\left(\frac{1-n}{n} - \frac{\cos n\pi}{n}, \pm \frac{\sin n\pi}{n}\right);$$

at these points $dx/d\phi = 0$, $\phi = 0$, and x as a function of ϕ has its maximum value.

The fluid is thus flowing into a diverging channel or spout with parallel walls inclined at an angle $2n\pi$ radians to each other. The opening is $(2 \sin n\pi)/n$ wide. The volume of fluid that flows out per second, per unit of length perpendicular to the xy-plane, is the total increment of ψ across the opening or 2π. Part of the flow net for $n = 1/4$ is shown in Figure 96.

If in the formulas ϕ and ψ are replaced by ϕ/k and ψ/k, respectively, $\psi = \pm \pi k$ on the walls and the volumetric rate of outflow is $2\pi k$. If $k < 0$, the flow is reversed. If the expressions given for z, x, and y are all multiplied by c, the opening is $(2c \sin n\pi)/n$ wide. Both changes may be made. Velocities are multiplied by k or by $1/c$, or, if both changes are made, by k/c.

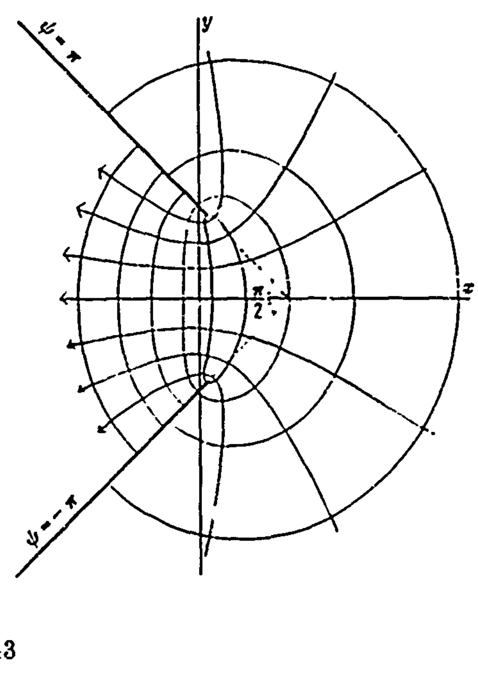
For $n = 1/2$, the spout becomes a slotted plate and the transformation reduces to a modified form of that in Section 61. For $1/2 < n < 1$, the transformation merely repeats itself with changes of scale, orientation, and direction of flow.

(For notation and method; see Section 34; Reference 1, Article 66.)

64. TWO-DIMENSIONAL PITOT TUBE

$$z = w + \ln w. \qquad [64a]$$

Figure 96 – Flow net for fluid entering a diverging spout. See Section 63. (Copied from Reference 255.)



This transformation may be obtained by superposing a uniform velocity upon the flow out of a spout having parallel walls in such manner as to reduce the fluid to rest within the depths of the spout. It is convenient first to reverse the flow through the spout by substitut- $-w_1$ for w in Equation [62a], which gives

$$z = -w_1 + e^{-w_1} \qquad [64b]$$

For uniform flow at unit velocity toward negative x, the complex potential is $w_2 = z$, with $\phi_2 = x$. Combining the two flows, the complex potential is $w = w_1 + w_2 = w_1 + z$. Substitution of $w_1 = w - z$ in Equation [64b] gives $z = -w + z + e^{-w+z}$ or $w = e^{-w+z}$, which is equivalent to Equation [64a].

Then, from $z = x + iy$, $w = \phi + i\psi$,

$$x = \phi + \tfrac{1}{2}\ln(\phi^2 + \psi^2), \qquad y = \psi + \tan^{-1}\frac{\psi}{\phi}, \qquad [64c,d]$$

$$-u + iv = \left(\frac{dz}{dw}\right)^{-1} = \frac{w}{1+w} = \frac{\phi + i\psi}{1 + \phi + i\psi},$$

$$u = -\frac{\phi + \phi^2 + \psi^2}{(1+\phi)^2 + \psi^2}, \qquad v = \frac{\psi}{(1+\phi)^2 + \psi^2} \qquad [64e,f]$$

Since y changes sign with ψ, symmetry exists with respect to the x-axis. Furthermore, the expression for y is many-valued, with a period of 2π. To make y single-valued, let $\tan^{-1}(\psi/\phi)$ have always the sign of ψ and be numerically less than π.

Assume that $\psi > 0$. Then, if ψ is large, so is y. Furthermore, along any streamline or curve for constant ψ, as ϕ ranges from $+\infty$ to $-\infty$, y continually increases, with a total increase of π, while x decreases on the whole from $+\infty$ to $-\infty$. If $\psi > \tfrac{1}{2}$,

$$\frac{\partial x}{\partial \phi} = 1 + \frac{\phi}{\phi^2 + \psi^2} > 0,$$

since the equation, $\phi^2 + \phi + \psi^2 = 0$ has no real roots for ϕ; hence x varies always in the same direction along the streamline. If $0 < \psi < \tfrac{1}{2}$, however, x retrogrades for a time as ϕ decreases, giving an S-shape to the streamline, as illustrated in Figure 97.

To locate the streamline for $\psi = 0$, keep ϕ constant and let $\psi \to 0$. Then, if $\phi > 0$, from [64c,d] $y \to \tan^{-1}(\psi/\phi) \to 0$, $x \to \phi + \ln\phi$. Hence, with $\psi = 0$, as ϕ varies from $+\infty$ to 0, x traces the entire x-axis and $\phi \to 0$ as $x \to -\infty$. If, however, $\phi < 0$, $y \to \tan^{-1}(\psi/\phi) \to \pm\pi$ and $x \to \phi + \ln(-\phi)$. Here $(\ln\phi^2)/2$ is written as $\ln(-\phi)$ rather than as $\ln\phi$ because $\phi < 0$ and the real logarithm is intended. Hence, as ϕ decreases from 0 to $-\infty$, x first increases from $-\infty$ to a maximum value of -1 at $\phi = -1$, where $dx/d\phi = 0$, and then decreases again to

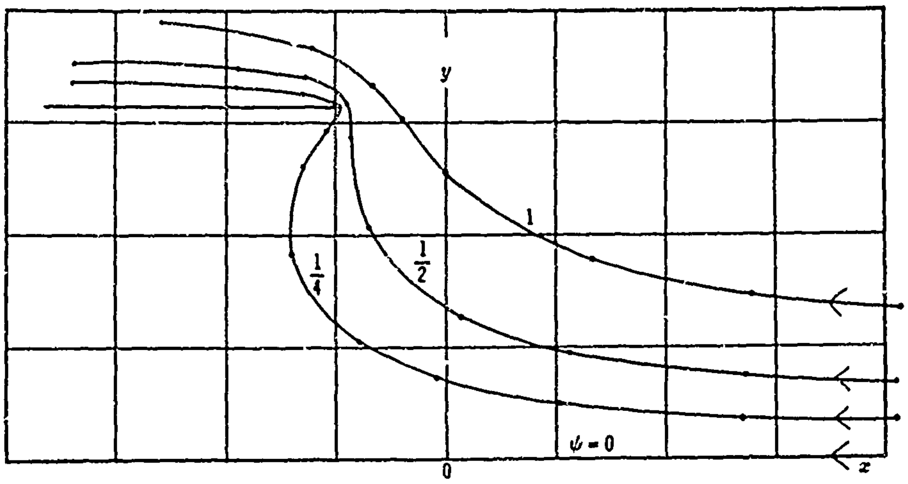


Figure 97 – A few streamlines for fluid flowing past a two-dimensional pitot tube. The x-axis is drawn along the median plane of the tube. (Copied from Reference 256.)

$-\infty$. For, a positive number always exceeds its logarithm, and toward infinity their ratio increases without limit. Furthermore, the lower half of the diagram is symmetrical with the upper. Hence, to sum up, the streamline for $\psi = 0$ follows the x-axis to $x = -\infty$, where $\phi = 0$, returns along both of the straight lines $y = \pi$ and $y = -\pi$ to $x = -1$, where $\phi = -1$, and then retraces these lines to $x = -\infty$, where $\phi = -\infty$.

Streamlines for a value of ψ close to 0 dip a certain distance into the space between the two lines and then emerge again.

Singularities occur only at $(-1, \pm\pi)$, where $\phi = -1$, $\psi = 0$ and $q \to \infty$. With the definition of $\tan^{-1}$ that has been adopted, however, the velocity is discontinuous across the lines $y = \pm\pi$, because of the discontinuity in ϕ. In the space between these lines, $q \to 0$ as $x \to -\infty$, since both ϕ and ψ then vanish; but elsewhere toward infinity $q \to 1$, since $|\phi| \to \infty$ or $|\psi| \to \infty$ or both, so that $dw/dz \to 1$.

In a physical case, therefore, boundaries must be inserted along the straight lines $x < -1$, $y = \pm\pi$. They form a two-dimensional pitot-tube with parallel plane walls 2π apart, placed in a stream of fluid approaching at unit velocity in a direction parallel to the walls. A few of the streamlines above the x-axis or median plane, labeled with values of ψ, are shown in Figure 97. Along the walls and also along the x-axis, $x = \phi + \ln|\phi|$, $q = |u|$, $u = -\dfrac{\phi}{1+\phi}$; on the walls $\phi < 0$, on the x-axis, $\phi > 0$.

The results may be generalized, as in Section 62, by replacing z, x, y, w, ϕ, ψ, in all formulas by z/c, x/c, y/c, gw, $g\phi$, $g\psi$. The tube is then $2\pi c$ wide, and the values of all velocities are divided by cg.

(For notation and method; see Section 34; Reference 1, Article 66.)

65. LAMINA BETWEEN WALLS

$$\sinh w = g \sinh z, \ g \text{ real and } g > 1. \qquad [65a]$$

$$\sinh \phi \cos \psi = g \sinh x \cos y, \ \cosh \phi \sin \psi = g \cosh x \sin y, \qquad [65b,c]$$

from $w = \phi + i\psi$, $z = x + iy$ and hyperbolic formulas in Section 32. The functions ϕ and ψ are many-valued. It is readily seen that continuous values can be chosen so as to satisfy the following description.

ϕ has the sign of x and ψ that of y; if $y = 0$, $\psi = 0$. Thus the x- and y-axes represent planes of symmetry. As $x \to \infty$, $\phi \to \infty$ also; furthermore, $\coth \phi \to \coth x \to 1$, so that, since from [65b,c] $\coth \phi \tan \psi = \coth x \tan y$, $\psi \to y$. For all values of x, $\psi = y$ on the lines $y = 0$, $y = \pi/2$, $y = \pi$; then one of Equations [65b,c] is satisfied automatically and the other fixes ϕ in terms of x. Furthermore, on the y-axis, wherever $\sin y > 1/g$, in [65c], $g \cosh x \sin y > 1$ and this equation can be satisfied only if $\phi > 0$; then, to make $x = 0$, $\cos \psi = 0$. In particular, $\cos \psi = 0$ and $\psi = \pi/2$ on the segment defined as follows:

$$\frac{\pi}{2} - \cos^{-1}\frac{1}{g} \leq y \leq \frac{\pi}{2} + \cos^{-1}\frac{1}{g}.$$

On this segment of the y-axis ϕ is discontinuous, since here $\cosh \phi = g \sin y > 1$ but ϕ must change sign with x as the y-axis is crossed.

For a physically possible case, a plane lamina must be inserted along the segment of the y-axis in question; and walls may also be inserted along the lines $y = 0$ and $y = \pi$. Then the flow is represented between these walls, with unit velocity at infinity where $\psi \to y$, past a lamina of width $L = 2 \cos^{-1} (1/g)$ placed perpendicular to the walls and midway between them; see Figure 98a.

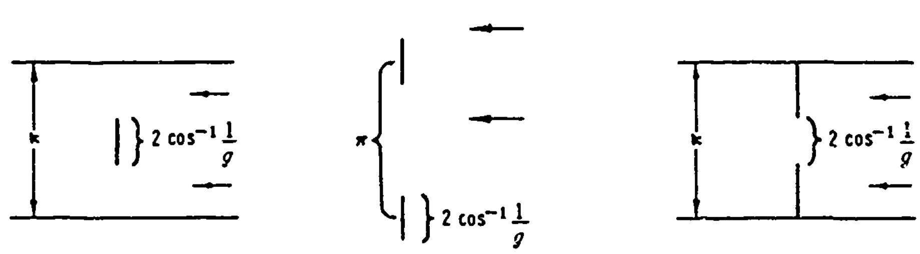


Figure 98 – (a) Lamina between walls, or (b) a grating of laminas; (c) slot in a partition between walls.

On the walls, $\psi = y$, $\sinh \phi = g \sinh x$. [65d]

On the median line, $y = \pi/2$, $\psi = \pi/2$, $\cosh \phi = g \cosh x$. [65e]

On the lamina, $\psi = \pi/2$, $\cosh \phi = g \sin y$. [65f]

In the plane containing the lamina but between it and the walls, $\phi = 0$, $v = 0$ and

$$\sin \psi = g \sin y, \qquad u = -\frac{\partial \psi}{\partial y} = -\frac{g|\cos y|}{\sqrt{1 - g^2 \sin^2 y}}. \qquad [65g,h]$$

Without the walls, the formulas may represent a stream falling perpendicularly upon a grating composed of such laminas lying in a common plane and spaced π apart; see Figure 98b.

The *similar transformation* $\cosh w = g \cosh z$ replaces the lamina between walls by an opening of width $2 \sin^{-1} (1/g)$ in a transverse partition between the walls; see Figure 98c.

Kinetic Energy of the Fluid

If the lamina moves in translation parallel to the walls at unit velocity, with the fluid at rest at infinity, the complex potential becomes $w - z$, where w is still given by [65a]. Substituting $w - z$ for w in Equation [76a], and $U = 1$, the kinetic energy of the fluid is

$$T_1 = \frac{1}{2} \rho \, (I') \oint (w - z)\, dz, \qquad [65i]$$

since for the lamina $S = 0$. Let the path of integration be displaced into a long rectangle with sides on the walls and ends at $x = \pm l$. This does not alter the value of the integral; see Section 29. On either wall, $dz = \pm x$ and $w - z = \phi - x$ since $\psi = y$; hence it is easily seen that the contributions of the walls to the integral cancel each other. Over each end, ϕ is practically constant and equal to its value at the corners, so that, when $x = l$ and l is large, from [65d], $\sinh \phi$ being positive $e^{\phi} = g e^{x}$, approximately and $\phi = x + \ln g$; whereas when $x = -l$, $\sinh \phi$ is negative, $e^{-\phi} = g e^{-x}$ and $\phi = x - \ln g$. Thus the integral over the two ends is

$$\int (w - z)\, dz = \int_0^{\pi} (\ln g + i\psi - iy)\, i\, dy + \int_{\pi}^{0} (-\ln g + i\psi - iy)\, i\, dy = 2 \int_0^{\pi} (\ln g)\, i\, dy = 2\pi i \ln g.$$

Hence

$$T_1 = \pi \rho \ln g.$$

Generalization

The distance between the walls, or between the centers of the laminas in the grating, may be changed from π to a, and also the velocity at infinity from unity to U, by substituting in all formulas $\pi z/a$, $\pi x/a$, $\pi y/a$, $\pi w/aU$, $\pi \phi/aU$, $\pi \psi/aU$, u/U for z, x, y, w, ϕ, ψ, u, respectively. Thereby all velocities are multiplied by U; the width of the lamina becomes

$$L = \frac{2a}{\pi} \cos^{-1} \frac{1}{g}; \qquad [65j]$$

and the kinetic energy of the fluid per unit length of the lamina is

$$T_1 = \frac{1}{\pi} \rho a^2 U^2 \ln g = \frac{1}{\pi} \rho a^2 U^2 \ln \sec \frac{\pi L}{2a} \qquad [65k]$$

If L/a is small, using the series for cos and ln from Section 33,

$$T_1 = \frac{\pi}{8} \rho U^2 L^2 \left(1 + \frac{\pi^2 L^2}{24a^2} \ldots .\right). \qquad [65l]$$

(For notation and method; see Section 34; Reference Love[50] and Taylor.[33]

66. LAMINAS OR CYLINDERS AND SURFACES

A lamina in other positions between rigid walls was studied, with reference to the lift when there is circulation around it, by Rosenhead[51] and Tomotika,[52] and more generally by Tomotika and others,[53–58], and by Havelock.[59] When the lamina is centered but inclined at an angle to the walls, with the circulation so chosen as to make the velocity finite at the trailing edge, the lift is increased by the presence of the walls, largely because the necessary circulation is itself increased.

Cylinders of a certain shape between walls, including a first approximation to a circular cylinder, are discussed in Sections 46 and 47. A plate near a single rigid wall was studied by Villat,[60] Raimondi,[61] and Tomotika and others,[62–65] and, with one edge on the wall, by Dätwyler[66] and Tomotika and Imai.[67] The wall increases the lift, at least at small angles of incidence. The effect of a neighboring free surface was studied by Tomotika and Imai.[68]

A circular arc near a rigid wall was treated by Jones,[69] and, for the case of actual contact, by Tomotika and Imai.[70] A cylinder near a wall was discussed in general terms by Villat.[60]

Circulatory flow between a cylinder and a free surface was treated by Vitali.[71]

CIRCULAR CYLINDERS

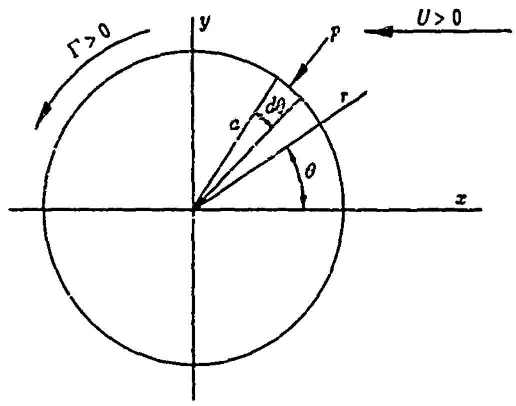
67. SYMMETRICAL FLOW PAST A CIRCULAR CYLINDER; DIPOLE IN A PARALLEL STREAM, OR INSIDE A COAXIAL SHELL

$$w = U\left(z + \frac{a^2}{z}\right), \text{ } U \text{ and } a \text{ real constants, } a > 0. \qquad [67a]$$

From $w = \phi + i\psi$ and $z = x + iy$,

$$\phi = U\left(1 + \frac{a^2}{r^2}\right)x, \quad \psi = U\left(1 - \frac{a^2}{r^2}\right)y, \quad r = (x^2 + y^2)^{1/2}, \qquad [67b,c,d]$$

Figure 99 – Diagram for flow past a circular cylinder.



or

$$\phi = U\left(r + \frac{a^2}{r}\right)\cos\theta, \qquad \psi = U\left(r - \frac{a^2}{r}\right)\sin\theta \qquad [67e,f]$$

in terms of polar coordinates r, θ, such that $x = r\cos\theta$ and $y = r\sin\theta$ as illustrated in Figure 99. The components of velocity are

$$u = U\left(-1 + a^2\,\frac{x^2 - y^2}{r^4}\right), \qquad v = \frac{2a^2\,xy}{r^4}, \qquad [67g,h]$$

or

$$q_r = U\left(-1 + \frac{a^2}{r^2}\right)\cos\theta, \qquad q_\theta = U\left(1 + \frac{a^2}{r^2}\right)\sin\theta, \qquad [67i,j]$$

$$q^2 = U^2\left(1 - \frac{2a^2}{r^2}\cos 2\theta + \frac{a^4}{r^4}\right). \qquad [67k]$$

There is a singularity at the origin, where $q \to \infty$. Stagnation points occur at $(a,0)$ and $(-a,0)$.

The x-axis represents a plane of symmetry for the flow, and the flow net has also a plane of geometrical symmetry along the y-axis.

At large distances $\phi \to Ux$ and the fluid is flowing toward negative x (if $U > 0$), or toward $\theta = \pi$, with uniform velocity U. The formulas represent, in fact, such a uniform stream superposed upon the flow due to a line dipole at the origin of dipole moment $\mu = a^2 U$, as is evident from formulas in Sections 35 and 37. The axis of the dipole is directed oppositely to the stream.

Along the x-axis $\psi = 0$, and $\psi = 0$ also on the circle $r = a$. This circle may be taken to represent a circular cylinder, and the formulas then represent a stream, uniform at infinity, flowing past such a cylinder. On the cylinder itself $\phi = 2\,Ux$, $q_r = 0$, and $q = |q_\theta|$ where

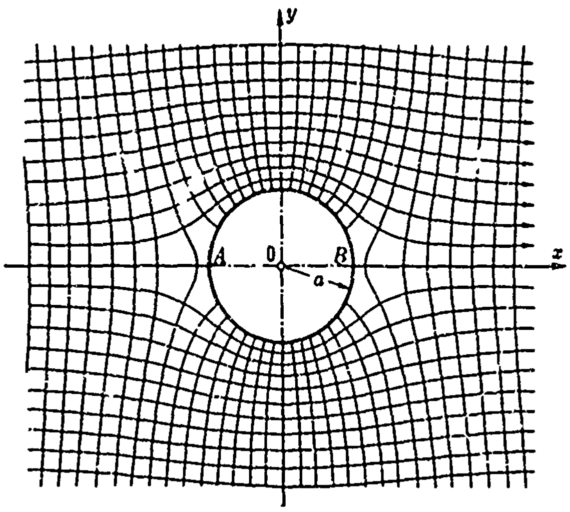


Figure 100 – Flow net for symmetrical flow past a circular cylinder. (Copied from Reference 8.)

$$q_\theta = 2\,U \sin\theta. \qquad [67l]$$

The flow net is shown in Figure 100; the points A and B represent the stagnation lines. The streamlines for $\psi = 0$ follow the plane of the x-axis to the stagnation line on the cylinder at $\theta = 0$, divide and proceed around both sides of the cylinder to the other stagnation line, then continue toward negative x.

If the motion is steady, the Bernoulli equation for the pressure p, assumed zero at infinity, gives on the cylinder itself

$$p = \tfrac{1}{2}\rho(U^2 - q^2) = \tfrac{1}{2}\rho\,U^2(1 - 4\sin^2\theta). \qquad [67m]$$

Thus $p = 0$ at $\theta = \sin^{-1}(\pm\tfrac{1}{2})$ or $\theta = \pm 30°$ or $\pm 150°$. At $\theta = 0°$ or $180°$, $p = \rho\,U^2/2$; at $\theta = 90°$ or $270°$, where $q = 2\,|U|$ and is a maximum, $p = -3\,\rho\,U^2/2$ and is a minimum. Because of the symmetry, there is no net force on the cylinder.

On the x-axis, where $\theta = 0$ or π,

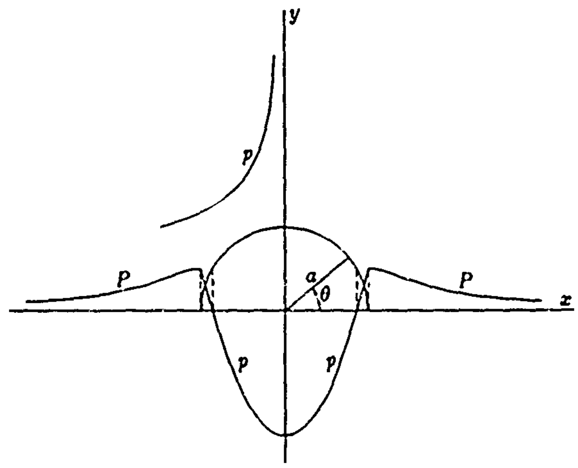
$$p = \tfrac{1}{2}\rho\,U^2\left(2\,\frac{a^2}{x^2} - \frac{a^4}{x^4}\right); \qquad [67n]$$

on the y-axis, where $\theta = 90°$ or $270°$,

$$p = -\tfrac{1}{2}\rho\,U^2\left(2\,\frac{a^2}{y^2} + \frac{a^4}{y^4}\right). \qquad [67o]$$

These formulas for p are plotted in Figure 101. The lower curve shows p along the x-axis and on the surface of the cylinder, plotted against x; the upper curve shows p at points on the y-axis outside of the cylinder, plotted horizontally with negative values to the left.

Figure 101 – Pressure in the symmetrical flow past a circular cylinder, along the axis of the flow, over the cylinder, and in the equatorial plane. See Equations [67m,n,o].



At points *inside* the cylindrical surface, the formulas may be used to represent the flow caused inside of a rigid cylindrical shell of radius a by a line dipole of moment $\mu = a^2 U$ on its axis. In this use of the formulas, U represents merely a constant having the value μ/a^2.

Changing the sign of U reverses all velocities, without affecting the flow net or the pressure.

(For notation and method, see Section 34; Reference 1, Article 68; Reference 2, Section 6.22, 6.23.)

68. TRANSLATION OF A CIRCULAR CYLINDER

By changing to a frame of reference that is moving toward negative x at velocity U, a description is obtained of a circular cylinder that is moving toward positive x at velocity U while the fluid is at rest at infinity. The change adds to w a term $-Uz$, representing uniform flow of the fluid toward positive x, so that, from [67a],

$$w = \frac{a^2 U}{z}, \qquad \phi = a^2 U \frac{\cos\theta}{r}, \qquad \psi = -a^2 U \frac{\sin\theta}{r}. \qquad \text{[68a,b,c]}$$

These formulas represent the dipole transformation, as discussed in Section 37. The axes of coordinates move here with the cylinder. The streamlines are arcs of circles, as illustrated in Figure 102.

The velocity components and the value of q^2 are:

$$u = a^2 U \frac{x^2 - y^2}{r^4}, \qquad v = a^2 U \frac{2xy}{r^4} \qquad \text{[68d,e]}$$

$$q_r = a^2 U \frac{\cos\theta}{r^2}, \qquad q_\theta = a^2 U \frac{\sin\theta}{r^2}; \qquad q^2 = \frac{a^4 U^2}{r^4}. \qquad \text{[68f,g,h]}$$

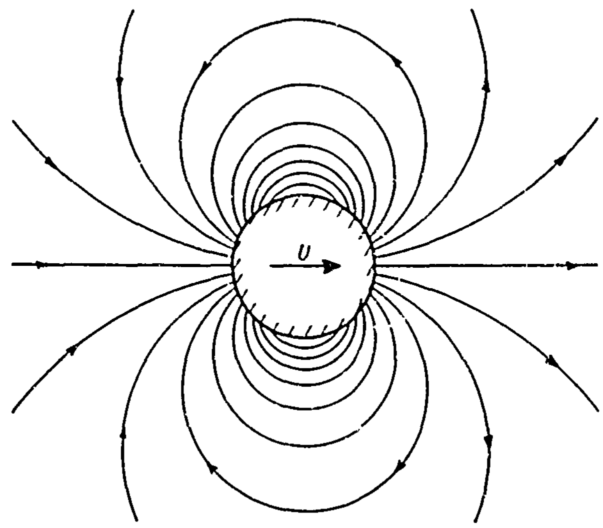


Figure 102 – Streamlines around a circular cylinder in translation. (Copied from Reference 1.)

Thus on the cylinder $q = |U|$. There are no stagnation points, but at the points $\theta = 0$ and $\theta = 180°$ the fluid is simply moving with the cylinder.

The distribution of pressure on the cylinder is the same as in the last section, and the net force on it vanishes if the motion is uniform.

The *kinetic energy* of the fluid per unit of length of the cylinder is

$$T_1 = \frac{1}{2}\rho \int_a^\infty dr \int_0^{2\pi} q^2\, r\, d\theta = \frac{\pi}{2}\, \rho\, a^2 U^2. \qquad [68i]$$

(For notation; see Section 34; Reference 1, Article 68; Reference 2, Section 9.20).

69. FLOW WITH CIRCULATION PAST A CIRCULAR CYLINDER

To introduce circulation around the cylinder, it is only necessary to add appropriate terms from Section 40. Then [67a,e,f] are replaced by

$$w = U\left(z + \frac{a^2}{z}\right) + \frac{i\Gamma}{2\pi} \ln \frac{z}{a}, \qquad z = re^{i\theta}, \qquad [69a,b]$$

$$\phi = U\left(r + \frac{a^2}{r}\right)\cos\theta - \frac{\Gamma}{2\pi}\,\theta, \qquad \psi = U\left(r - \frac{a^2}{r}\right)\sin\theta + \frac{\Gamma}{2\pi} \ln\frac{r}{a}. \qquad [69c,d]$$

Here r, θ are polar coordinates with origin on the axis of the cylinder, whose radius is a, and with θ measured from the positive x-axis, as in Figure 99; U and Γ are real constants.

The velocity components are

$$q_r = U\left(-1 + \frac{a^2}{r^2}\right)\cos\theta, \qquad q_\theta = U\left(1 + \frac{a^2}{r^2}\right)\sin\theta + \frac{\Gamma}{2\pi r} \qquad [69e,f]$$

Thus at infinity $q_r = -U\cos\theta$ and $q_\theta = U\sin\theta$, so that the flow is parallel to the x-axis and directed toward negative x if $U > 0$. On the cylinder $q = |q_\theta|$ and $q_\theta = 2U\sin\theta + \Gamma/(2\pi a)$.

The constant Γ represents the circulation around any closed curve encircling the cylinder once in the positive direction, or in the direction of increasing θ. For ϕ, like θ, is many-valued; ϕ decreases by Γ in going around the cylinder.

The streamlines for $\Gamma = 0.6\,(4\pi aU)$ are illustrated in Figure 103, and for $\Gamma = 6\pi aU$ in

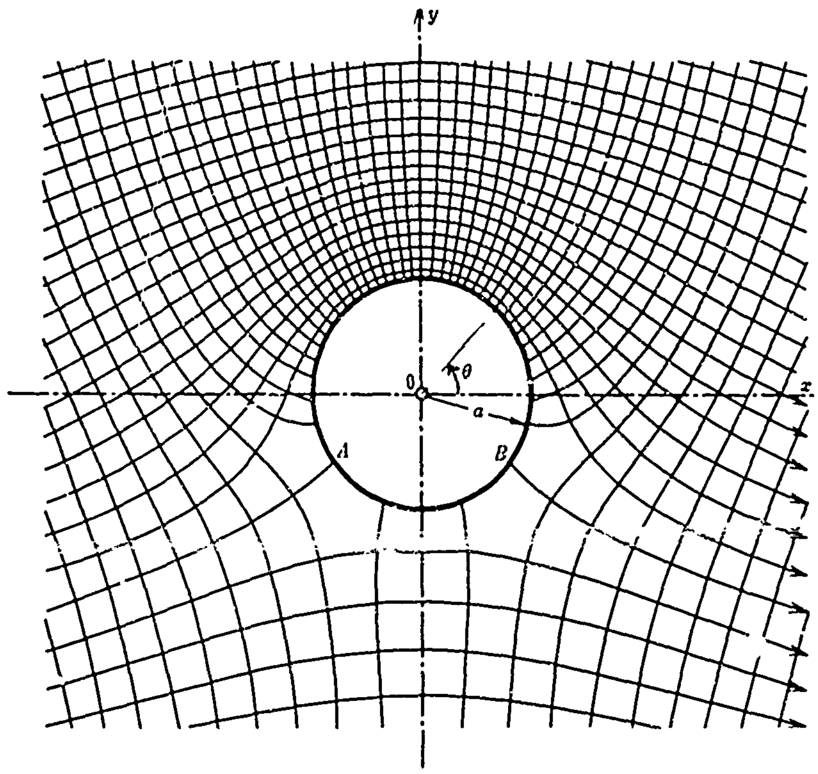


Figure 103a

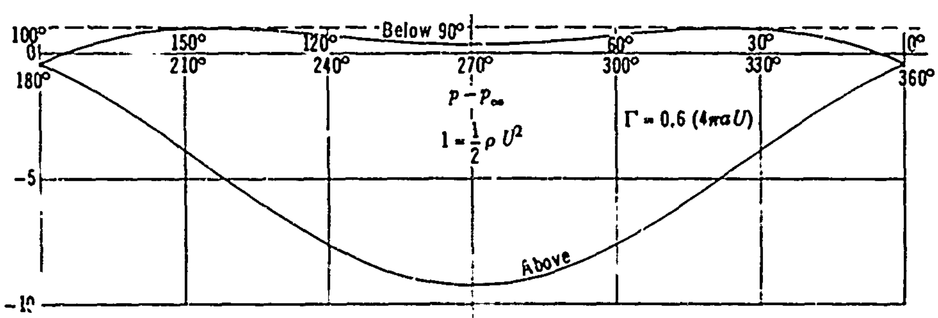


Figure 103b

Figure 103 -- Flow net for a stream with circulation past a circular cylinder, and distribution of pressure over the cylinder, plotted against θ. See Section 69. (The flow net is copied from Reference 8.)

Figure 104. The excess pressure on the cylinder, $p - p_\infty$, for steady motion, is shown in each case in terms of $\rho U^2/2$ as a unit. The angle θ is plotted as abscissa toward the left from 0 to π, then toward the right; "above" and "below" refer to the upper and lower halves of the cylinder as drawn.

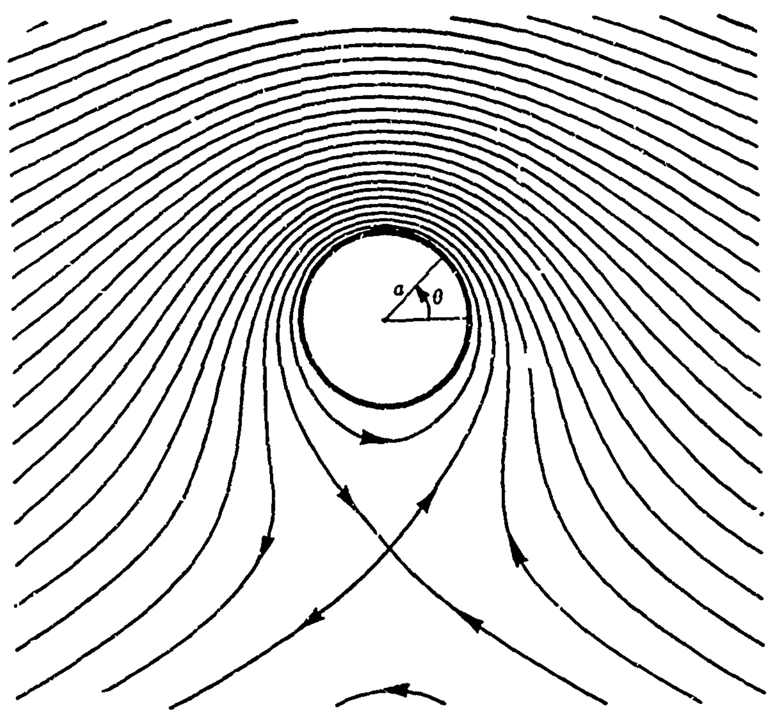


Figure 104a

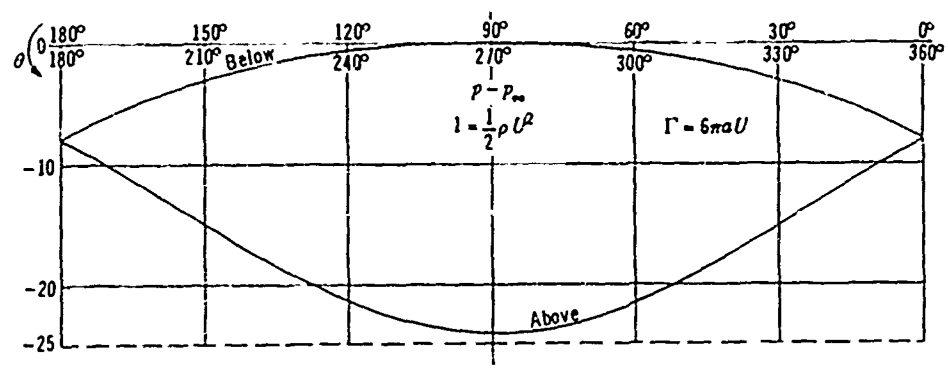


Figure 104b

Figure 104 – Streamlines for flow with stronger circulation past a circular cylinder, and distribution of pressure over the cylinder. See Section 69. (The streamlines are copied from Reference 257.)

Stagnation points occur where $q_r = 0$ and $q_\theta = 0$. The equation $q_r = 0$ is solved either by $r = a$ or by $\cos\theta = 0$. If the solution $r = a$ is possible, stagnation points occur on the cylinder at positions such that, to make $q_\theta = 0$,

$$\sin\theta = -\frac{\Gamma}{4\pi a U} \qquad [69g]$$

They are at A, B, in Figure 103. From [69g] and [69f] it is easily seen that the presence of circulation shifts both of the stagnation points toward the side on which it reduces the fluid velocity.

Suppose that Γ and U have the same sign. Then, if $\Gamma = 4\pi a U$, the two stagnation points come together at $\theta = -90°$. If $\Gamma/U > 4\pi a$, the equation for $\sin\theta$ cannot be solved; but now it is possible to assume that $\cos\theta = 0$, $\sin\theta = -1$, and to solve the equation $q_\theta = 0$ for r. Thus as Γ/U is increased above $4\pi a$ the stagnation point moves out along the radius $\theta = -90°$, occurring at

$$r = \frac{\Gamma}{4\pi U}\left(1 + \sqrt{1 - \frac{16\pi^2 a^2 U^2}{\Gamma^2}}\right) \qquad [69h]$$

The streamline that passes through such a stagnation point cuts itself perpendicularly and encircles the cylinder, as in Figure 104; all fluid inside it remains inside, circling round the cylinder along closed paths.

Changing the sign of Γ reverses the flow pattern and alters all velocities as if by reflection in a mirror along the axis $\theta = 0$ or π. Changing the signs of both Γ and U, however, merely reverses all velocities without other change.

If the motion is steady, so that the pressure is given by the Bernoulli equation, it is easily seen from symmetry that the resultant force on the cylinder is a force transverse to the direction of the stream, or a lift. Drag, in actual fluids, is an effect of viscosity, which is here assumed to be absent. On an element of area of width $a\,d\theta$ and of unit length in the direction of the axis, the force is $pa\,d\theta$, directed toward the axis. Hence the total force in the direction $\theta = \pi/2$ on unit length of the cylinder is, substituting $-\rho q^2/2$ for p and the value of q_θ^2 for q^2,

$$L = -\int_0^{2\pi}\left(-\frac{1}{2}\rho q^2\right) a \sin\theta\, d\theta = \rho\Gamma U \qquad [69i]$$

Here $-\sin\theta$ is introduced in taking a component of the force.

If the velocity at infinity makes an angle γ with the negative x-axis, and if the axis of the cylinder is displaced to the point $z = z_1 = x_1 + iy_1$, then

$$w = U\left[(z - z_1)\, e^{-i\gamma} + \frac{a^2}{z - z_1} e^{i\gamma}\right] + \frac{i\Gamma}{2\pi} \ln \frac{z - z_1}{a}, \qquad [69j]$$

$$\phi = U\left(r + \frac{a^2}{r}\right)\cos(\theta - \gamma) - \frac{\Gamma}{2\pi}\,\theta, \quad \psi = U\left(r - \frac{a^2}{r}\right)\sin(\theta - \gamma) + \frac{\Gamma}{2\pi} \ln \frac{r}{a}, \qquad [69k,l]$$

$$q_r = U\left(-1 + \frac{a^2}{r}\right)\cos(\theta - \gamma), \quad q_\theta = U\left(1 + \frac{a^2}{r^2}\right)\sin(\theta - \gamma) + \frac{\Gamma}{2\pi r}, \qquad [69m,n]$$

where $z - z_1 = re^{i\theta}$, $r = [(x - x_1)^2 + (y - y_1)^2]^{1/2}$,

and θ is a polar angle about the point z_1, or $\theta = \tan^{-1}[(y - y_1)/(x - x_1)]$. For, it is obvious that the entire flow is displaced in the desired manner; and at infinity the term $Uze^{-i\gamma}$ predominates in w and represents uniform flow at the angle γ, as in Section 35.

(For notation and method; see Section 34; Reference 1, Article 69; Reference 2, Section 7.12.)

70. TRANSLATION OF A CIRCULAR CYLINDER WITH CIRCULATION

By viewing the situation discussed in the last section from a frame of reference that moves with the fluid at infinity, as in Section 68, formulas are obtained for the case of a cylinder that has circulation around it but is moving toward positive x at velocity U, while the fluid is at rest at infinity. For this case

$$w = \frac{a^2 U}{z} + \frac{i\Gamma}{2\pi} \ln \frac{z}{a}, \qquad [70a]$$

$$\phi = a^2 U \frac{\cos\theta}{r} - \frac{\Gamma}{2\pi}\theta, \quad \psi = -a^2 U \frac{\sin\theta}{r} + \frac{\Gamma}{2\pi} \ln \frac{r}{a}, \qquad [70b,c]$$

$$q_r = a^2 U \frac{\cos\theta}{r^2}, \quad q_\theta = a^2 U \frac{\sin\theta}{r^2} + \frac{\Gamma}{2\pi r} \qquad [70d,e]$$

The axes of coordinates move with the cylinder.

A stagnation point can now occur only where $\cos\theta = 0$ and also

$$-\frac{r\Gamma}{2\pi a^2 U} = \sin\theta = \pm 1. \qquad [70f]$$

For $\Gamma/U > 0$, $\theta = -90°$; for $\Gamma/U < 0$, $\theta = 90°$; and in either case

$$r = 2\pi a^2 \left|\frac{U}{\Gamma}\right|. \qquad [70g]$$

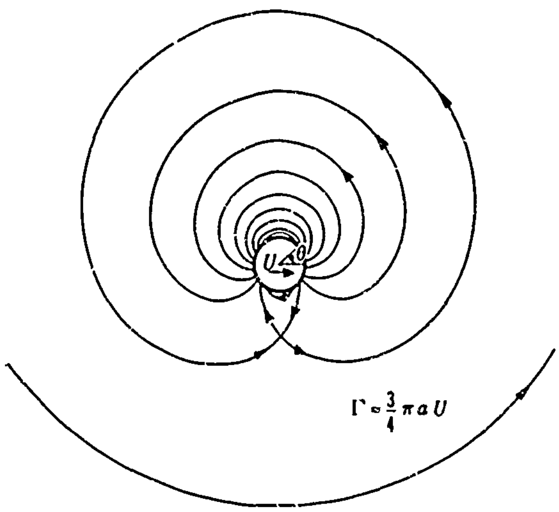


Figure 105 – Streamlines around a circular cylinder in translation and with circulation around it; the fluid is at rest at infinity. See Section 70. (Copied from Reference 1.)

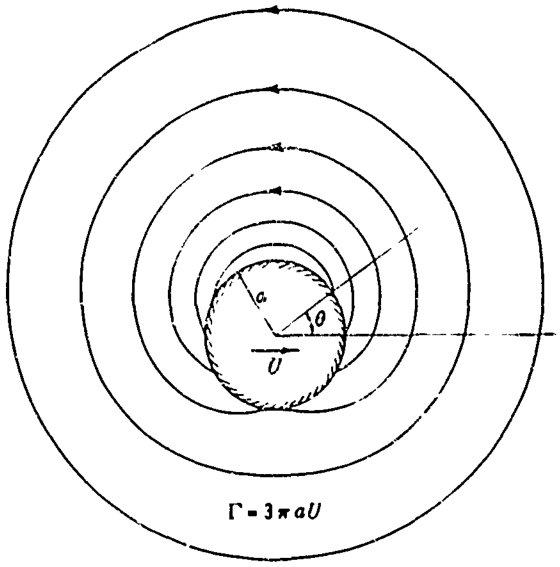


Figure 106 – Same as Figure 105 but with the circulation four times as strong.

Since, however, it is necessary that $r \geqq a$, a stagnation point can occur only if $|\Gamma| \leqq 2\pi a|U|$. If $\Gamma = 2\pi aU$, it lies on the surface of the cylinder.

Streamlines for $\Gamma = 3\pi aU/4$ and $\Gamma = 3\pi aU$ are shown in Figures 105 and 106.

The pressure and the lift on the cylinder are as in the last section.

Changing the sign of Γ changes the diagram as if by reflection in the x-axis. Changing the signs of both Γ and U merely reverses all velocities.

If the direction of motion of the cylinder makes an angle γ with the positive x-axis, from [69j] with $z_1 = 0$,

$$w = \frac{a^2 U}{z} e^{i\gamma} + \frac{i\Gamma}{2\pi} \ln \frac{z}{a}, \qquad [70h]$$

$$\phi = \frac{a^2 U}{r} \cos(\theta - \gamma) - \frac{\Gamma}{2\pi} \theta, \qquad \psi = -\frac{a^2 U}{r} \sin(\theta - \gamma) + \frac{\Gamma}{2\pi} \ln \frac{r}{a}; \qquad [70i,j]$$

$$q_r = \frac{a^2 U}{r^2} \cos(\theta - \gamma), \qquad q_\theta = \frac{a^2 U}{r^2} \sin(\theta - \gamma) + \frac{\Gamma}{2\pi r}. \qquad [70k,l]$$

(Reference 1, Article 69; Reference 2, Section 9.60.)

71. CYLINDER AND VORTICES IN A STREAM

(1) A single vortex. Let a cylinder of radius a be stationary in a stream approaching at velocity U toward negative x; let there be circulation Γ about the cylinder, and also a line vortex with circulation Γ_1 located at the external point $z = b = -he^{i\gamma}$, or $(-h \cos \gamma, -h \sin \gamma)$, the origin being taken on the axis of the cylinder. Here h and γ are real, and the vortex is located on a radial line inclined at an angle γ to a radius drawn in the direction of the stream; see Figure 107. This case is of interest in the theory of wakes.

The complex potential representing the partial flow caused by the vortex is given by Equation [42a] with A replaced by $\Gamma_1/2\pi$, $z - c$ by $z - b$ or $z + he^{i\gamma}$, and $z + c$ by $z - b'$ where $b' = -(a^2/h)e^{i\gamma}$; for, as shown in Section 42(B), the image vortex lies on the inverse line with respect to the cylinder and hence on the same radial line as the actual vortex but at a distance a^2/h from the axis of the cylinder. In this flow the circulation around the cylinder is $-\Gamma_1$. Circulation $\Gamma + \Gamma_1$ must then be added in order to make the total equal to Γ. This can be accomplished, in superposing the flow due to the stream, by using Equation [69a] with Γ replaced by $\Gamma + \Gamma_1$. The complete complex potential thus constructed is

$$w = U\left(z + \frac{a^2}{z}\right) + \frac{i(\Gamma + \Gamma_1)}{2\pi} \ln \frac{z}{a} + \frac{i\Gamma_1}{2\pi} \ln \frac{z - b}{z - b'} \qquad [71a]$$

Expressions for ϕ, ψ, and the velocity are easily derived if needed.

The *force* on the cylinder may be found from the Blasius theorem, Equation [74g], provided the vortex is assumed to remain stationary. Here

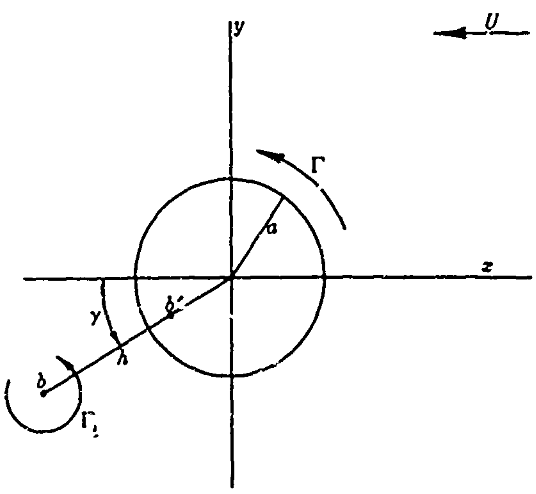


Figure 107 – A line vortex at "b" near a circular cylinder in a stream.

$$\frac{dw}{dz} = U\left(1-\frac{a^2}{z^2}\right)+\frac{i(\Gamma+\Gamma_1)}{2\pi}\frac{1}{z}+\frac{i\Gamma_1}{2\pi}\left(\frac{1}{z-b}-\frac{1}{z-b'}\right). \qquad [71b]$$

The force is obtained from the residues at $z = 0$ and $z = b'$; compare Sections 30 and 42. It is found that

$$X = \frac{\rho\Gamma_1}{2\pi}\left(\frac{\Gamma+\Gamma_1}{h}-\frac{h\Gamma_1}{h^2-a^2}\right)\cos\gamma - \frac{a^2}{h^2}\rho\Gamma_1 U\sin 2\gamma, \qquad [71c]$$

$$Y = \rho\Gamma U + \frac{\rho\Gamma_1}{2\pi}\left(\frac{\Gamma+\Gamma_1}{h}-\frac{h\Gamma_1}{h^2-a^2}\right)\sin\gamma + \frac{a^2}{h^2}\rho\Gamma_1 U\cos 2\gamma. \qquad [71d]$$

A negative value of x represents a force in the direction of the stream or a drag.

If $\Gamma = -\Gamma_1$, the formulas are simplified. Streamlines for such a case are shown in Figure 108; here Γ is positive and the flow is from right to left.

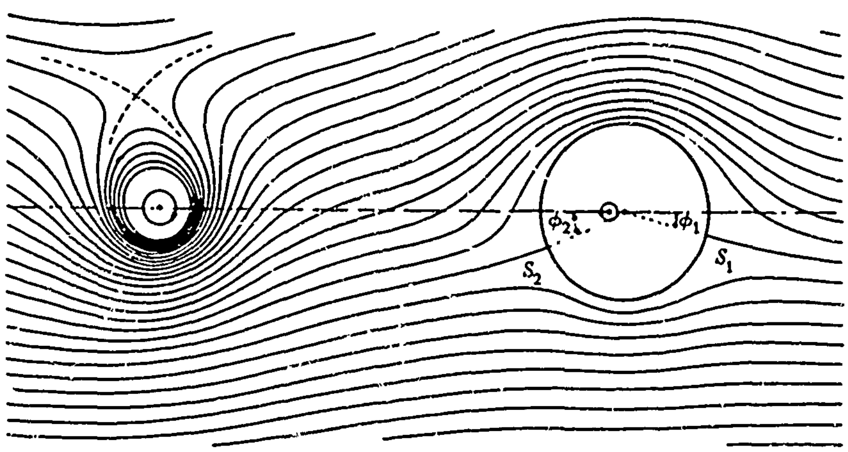


Figure 108 – Streamlines near a circular cylinder due to a stream with circulation Γ about the cylinder and a line vortex of circulational strength $-\Gamma$. (Copied from Reference 28.)

(2) Several vortices. If several vortices are present, the forces due to them are simply additive; each term in X and Y that contains Γ_1 is replaced by a sum of such terms, one for each vortex. In Figure 109 are shown streamlines for a symmetrical case with $\Gamma = 0$ and two vortices. (References Müller[28], Bickley[72], and Morris[73].)

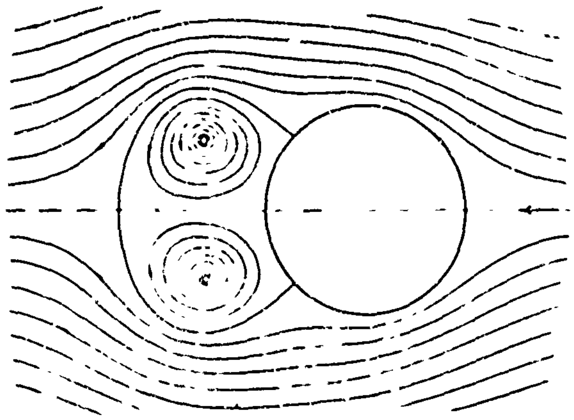


Figure 109 – Symmetrical streamlines near a circular cylinder caused by a stream and two equal and opposite vortices. (Copied from Reference 28.)

FORCES ON CYLINDERS

72. THE DISTANT MOTION

The fluid at infinity is usually assumed to be either at rest or in uniform motion. Sometimes it is of interest to know how rapidly this condition is approached.

Suppose, for generality, that at infinity the fluid is moving uniformly at speed U, and that the motion is irrotational everywhere except perhaps inside a certain cylindrical surface S, which may enclose within it one or more solid cylinders. The cross section of S need not be circular. Then at large distances from S a closer approximation to the actual motion can be secured by superposing the following three motions:

(a) The uniform flow at speed U;

(b) A motion in which the velocity is everywhere directed along a radius from any chosen axis inside S and is of magnitude

$$q_v = \frac{V}{2\pi r}; \qquad [72a]$$

(c) A circulatory motion in which the velocity is perpendicular to the radius drawn from the chosen axis and of magnitude

$$q_\Gamma = \frac{\Gamma}{2\pi r}. \qquad [72b]$$

Here r denotes distance from the axis,

V is the net volume per second that flows outward across S, per unit of its length, due to sources or sinks inside it, and

Γ is the circulation along any path encircling S once in the positive direction.

V, q_v, q_Γ, and Γ may be positive or negative.

To establish this result, let the potential ϕ for the actual motion be written as the sum of the potential for the approximate motion thus constructed and a fourth component ϕ', or

$$\phi = \phi_U - \frac{V}{2\pi} \ln r - \frac{\Gamma}{2\pi} \theta + \phi'$$

where ϕ_U is the potential for the uniform flow; see Section 40. The partial motion represented by ϕ' will then be one in which the net outflow or inflow across S vanishes, and in which the circulation likewise vanishes around every closed curve that does not cut S. Hence, in particular, ϕ' is singled valued, and so is the corresponding stream function ψ'. In such a motion, it can be shown that the velocity vanishes at infinity at least as fast as $1/r^2$, as it does, for example, in the symmetrical flow caused by a moving circular cylinder. At large distances from S the motion represented by ϕ' may accordingly be disregarded in comparison with the other components, and only motions (a), (b), and (c) remain.

The corresponding theorem for the complex potential is that, if du/dz is differentiable and single valued outside a curve S, and also finite at infinity, then, at sufficiently great distances from S, dw/dz can be expanded in a Laurent series of the form,

$$\frac{dw}{dz} = \ldots \frac{b_3'}{z^3} + \frac{b_2'}{z^2} + \frac{b_1'}{z} + a_0', \qquad [72c]$$

as stated in Section 27. Hence, by integration,

$$w = \ldots \frac{b_2}{z^2} + \frac{b_1}{z} + a_0 + c \ln z + a_1 z \qquad [72d]$$

where $\ldots b_2, b_1, a_0, c$ are constants, real or complex. The last two or three terms of this series represent the approximation just described; the real part of c gives a term representing the sources, the imaginary part a term representing the circulatory motion. The term in $\ln z$ is many valued.

If the surface S represents a rigid cylinder, there can be no source and c must be purely imaginary. Furthermore, the largest term that depends upon the shape and motion of S is the term b_1/z, which, as in Section 37, represents a dipole. Thus at large distances the effect of a moving cylinder with no circulation around it is that of a line dipole located in its neighborhood.

It is of interest, finally, to consider the effect of a conformal transformation upon the distant motion. In an important class of single-valued transformations the plane is left unaltered at infinity. Such a transformation from z to z' can be written, toward infinity, in the form

$$z = z' + \frac{b_1'}{z'} + \frac{b_1''}{z'^2} \; \ldots\ldots \qquad [72e]$$

Then, substituting in (72d), using the binomial theorem and also the series for $ln\,(1 + x)$ after writing $\ln z = \ln z' + \ln\,[1 + (z - z')/z']$,

$$w = \ldots\ldots \frac{b_1 + a_1 b_1'}{z'} + a_0 + c \ln z' + a_1 z'. \qquad [72f]$$

Thus the equivalent dipole at infinity is in general changed, in correspondence with the change in the shape and size of the cylinder. The circulation and source strength remain unchanged.

73. LIFT ON A CYLINDER; THE KUTTA-JOUKOWSKI THEOREM

In Section 70 it was shown that a circular cylinder moving through fluid otherwise at rest experiences no drag or force opposing its motion, but, if circulation is present, there is a transverse force or lift of magnitude

$$L = \rho \Gamma U \qquad [73a]$$

per unit of length of the cylinder. Here ρ is the density of the fluid, U is the velocity of translation of the cylinder perpendicularly to its length, and Γ is the circulation in the fluid around any closed path encircling it once. The fluid is assumed, as usual, to be incompressible and devoid of viscosity. The direction of the lift can be found by rotating the direction of motion of the cylinder through 90 deg in the direction of rotation suggested by Γ, as illustrated in Figure 110.

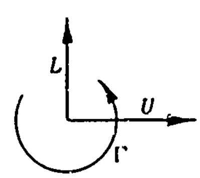


Figure 110 – Illustrating direction of the lift on a cylinder.

It was shown by Kutta and independently by Joukowski that the same statements are true for a cylinder of any form. This may be shown by considering the changes in the momentum of the fluid as the cylinder passes.

Let a frame of reference be used relative to which the cylinder is at rest while the fluid at infinity is moving with velocity $-U$. Consider the mass of fluid that lies, at any time t_1, between two planes drawn perpendicular to the axis of the cylinder and unit distance apart, and also between two other planes PP_1, $P'P_1'$, drawn perpendicular to the direction of

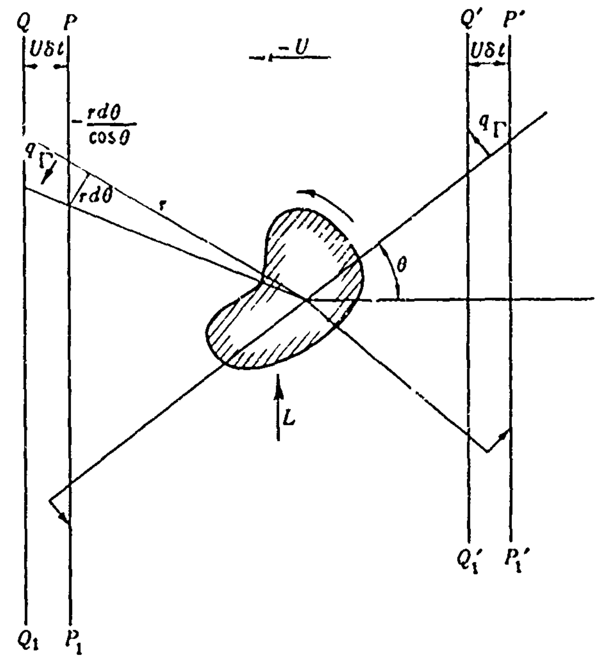


Figure 111 – Diagram for calculation of the lift.

U and far removed from the cylinder; see Figure 111. In time δt the boundaries of this fluid are displaced through a distance $U\,\delta t$ into new positions QQ_1, $Q'Q_1'$. At the end of the displacement, the part of the fluid that lies between PP_1, and $Q'Q_1'$ has the same momentum as had the part of the fluid that occupied this position originally, since the motion is steady. Hence this region may be disregarded; and the net change in momentum of the mass of fluid under discussion is equal to the momentum present in the newly occupied layer of space QQ_1PP_1 minus the momentum that has disappeared from the layer $Q'Q_1'P'P_1'$ which has been vacated.

In calculating the momentum in these two layers, use may be made of the approximate description of the motion given in the last section. Here $V = 0$, since there is no outflow from the cylinder. The uniform motion at velocity $-U$ contributes nothing to the difference in momentum between the two layers. The velocity q_Γ gives rise, on the whole, to no momentum having the direction of U, because of symmetry, but it does give rise to transverse momentum.

Take the axis of polar coordinates in the direction of U. Then an element of cross-sectional area of the layer $QP\,Q_1P_1$ is a parallelogram of height $U\,\delta t$ on a base of $rd\,\theta/(-\cos\,\theta)$, as in Figure 111, and, using [72b] for q_Γ, the transverse momentum in this layer is

$$-\int_{\pi/2}^{3\pi/2} \rho\left(-\frac{\Gamma}{2\pi r}\cos\,\theta\right) U\,\delta t\,\frac{r\,d\,\theta}{\cos\,\theta} = \frac{1}{2}\,\rho\Gamma U\delta t.$$

Equal but oppositely directed transverse momentum was initially present in the layer $Q'Q_1'P'P_1'$. The net change in transverse momentum of the original mass of fluid is thus twice the momentum in the first layer or $\rho\Gamma U\delta t$, in the downward direction in Figure 111. By the law of action and reaction, this must equal the opposite momentum given to the cylinder, or $L\delta t$. Equation [73a] follows; and the direction of L is easily seen to be as stated.

74. THE BLASIUS THEOREM.

This theorem provides useful formulas for the force on a cylinder of any shape, and also for the torque or moment of force, in the case of steady two-dimensional motion.

Consider an element of the surface of the cylinder which has a width ds and unit length in a direction perpendicular to the xy-plane, or to the planes of motion. Let the tangent to ds, drawn in the counterclockwise direction around the cylinder, make an angle θ with the x-axis, as shown in Figure 112. Then the force on the element due to the pressure p, taken positive when directed toward the interior of the cylinder, has a magnitude pds and Cartesian components

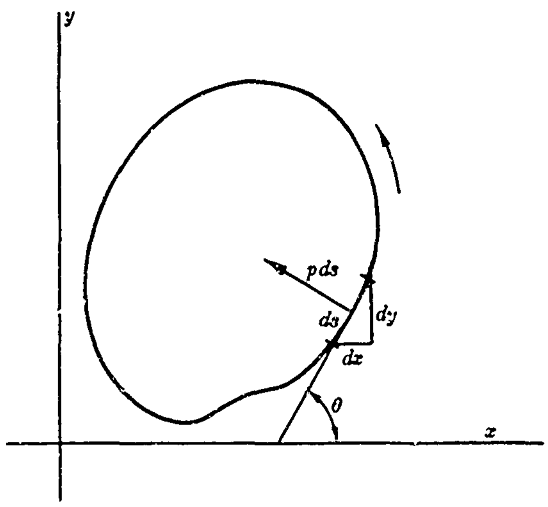


Figure 112 – Force on element ds of the surface of a cylinder due to a pressure p.

$$dX = -p\,ds\sin\theta, \quad dY = p\,ds\cos\theta$$

But $ds\cos\theta = dx$, $ds\sin\theta = dy$, where dx and dy are the components of ds. Hence

$$X = -\int p\,dy, \quad Y = \int p\,dx \qquad [74a, b]$$

Also, by multiplying each component of force by its lever arm, the total moment L about an axis passing through the origin is similarly found to be

$$N = \int p(x\,dx + y\,dy) = \int p\,r\,dr \qquad [74c]$$

where $r^2 = x^2 + y^2$.

These formulas are general. If, however, the motion is steady, then, the pressure being taken as zero at points where the velocity q is zero, $p = -\rho q^2/2$ where ρ is the density of the fluid and

$$X = \frac{1}{2}\rho\int q^2 dy,\quad Y = -\frac{1}{2}\rho\int q^2 dx,\quad N = -\frac{1}{2}\rho\int q^2 (x\,dx + y\,dy) \qquad [74d, e, f]$$

It was shown by Blasius that these expressions could be transformed so as to contain only integrals of a certain analytic function of z where $z = x + iy$; the methods of complex-variable theory then become available for their evaluation. From Equation [74d, e], since $q^2 = u^2 + v^2$,

$$X - iY = \frac{i}{2}\rho\int (u^2 + v^2)(dx - i\,dy)$$

Now $(u^2 + v^2)(dx - i\,dy) = (-u + iv)(-u - iv)(dx - i\,dy)$. Since the path of integration is part of a streamline, the vector (dx, dy) is parallel to the velocity or to (u, v) at the same point; hence $dy/dx = v/u$ or $v\,dx = u\,dy$. Thus

$$(-u - iv)(dx - i\,dy) = -u\,dx + iu\,dy - iv\,dx - v\,dy = (-u + iv)(dx + i\,dy),$$

and, using $-u + iv = dw/dz$, as in Equations [25i] and [34f], and $dz = dx + i\,dy$,

$$X - iY = \frac{i}{2}\rho\int \left(\frac{dw}{dz}\right)^2 dz \qquad [74g]$$

The torque requires a somewhat different artifice. Clearly

$$q^2 (x\,dx + y\,dy) = (R)\,[(u^2 + v^2)(x + iy)(dx - i\,dy)]$$

where the symbol (R) signifies that only the real part of what follows is to be taken. The same changes as before can be made in this expression. It is then found that, since the real part of any integral arises exclusively from the real part of the integrated expression, [74f] may be written

$$N = -\frac{1}{2}\rho\,(R)\int \left(\frac{dw}{dz}\right)^2 z\,dz \qquad [74h]$$

The integrals in these equations can be evaluated either for the entire contour of a stationary body or for any part of it, or even for part of a streamline in the fluid. In any case X and Y stand for the components of the total force transmitted toward the left across the chosen path of integration, and L for its moment about an axis through the origin.

The new formulas are especially advantageous when the path of integration is closed. Then, if dw/dz is given by a mathematical function that is analytic on the path, and also throughout its interior except for a few singular points, the integrals are given at once by the sum of the residues of the integrand at the singular points. It does not matter if these points actually lie in a region devoid of fluid.

As an example, the Blasius theorem may be used to prove the Kutta-Joukowski theorem. Under the conditions specified in Section 72, with the cylinder stationary, dw/dz is a regular function everywhere outside of the cylinder; hence, by the Cauchy integral theorem, the path of integration may be displaced toward infinity in all directions. As before, let the motion be steady; and at infinity let the fluid approach at speed U from a direction making an angle γ with the positive x-axis, so that its components of velocity are $-U\cos\gamma$, $-U\sin\gamma$. Then, in view of the results in Sections 35 and 72, w can be written for large z in the form

$$w = Ue^{-i\gamma}z + \frac{i\Gamma}{2\pi}\ln z + a_0 + \frac{b_1}{z} + \frac{b_2}{z^2}\cdots\cdots \qquad [74i]$$

where Γ is the circulation about the cylinder. The origin may be located at any finite point. Then, in powers of $1/z$,

$$\frac{dw}{dz} = Ue^{-i\gamma} + \frac{i\Gamma}{2\pi z} - \frac{b_1}{z^2} - \frac{2b_2}{z^3}\cdots\cdots \qquad [74j]$$

$$\left(\frac{dw}{dz}\right)^2 = U^2e^{-2i\gamma} + \frac{i\Gamma U}{\pi z}e^{-i\gamma} - \left(\frac{\Gamma^2}{4\pi^2} + 2b_1Ue^{-i\gamma}\right)\frac{1}{z^2}\cdots\cdots$$

Upon substituting this series for dw/dz in Equation [74g] and noting that $\oint dz/z^n = 2\pi i$ if $n = 1$ but $= 0$ for other integral values of n, as shown in Section 30, it is found that

$$X - iY = \frac{i}{2}\rho\left(\frac{i}{\pi}\Gamma Ue^{-i\gamma}\right)\oint\frac{dz}{z} = -i\rho\Gamma Ue^{-i\gamma}.$$

Hence

$$X = -\rho\Gamma U\sin\gamma, \quad Y = \rho\Gamma U\cos\gamma. \qquad [74k,l]$$

The magnitude of the force per unit length is thus $(X^2 + Y^2)^{1/2} = \rho\Gamma U$.

For the torque, Equation [74h] gives similarly

$$N = \frac{1}{2}\, \rho\, (R) \left\{ \left(\frac{\Gamma^2}{4\pi^2} + 2 b_1 U e^{-i\gamma} \right) \oint \frac{dz}{z} \right\}$$

Write

$$b_1 = b_1' + i b_1'', \quad b_1' \text{ and } b_1'' \text{ real.}$$

Then

$$N = 2\pi\rho U\, (R)\, (i b_1 e^{-i\gamma}) = 2\pi\rho U\, (b_1' \sin\gamma - b_1'' \cos\gamma). \qquad [74m]$$

Thus the torque is independent of Γ.

If the fluid is brought to rest at infinity by using a frame of reference moving with the fluid, the cylinder is moving at velocity U, with components $U \cos\gamma$, $U \sin\gamma$, but both force and torque are the same as before.

The final remark may be made that, if dw/dz contains two or more poles inside the path of integration, representing line sources, vortices, dipoles, or other singularities, these included singularities in combination with themselves or each other contribute nothing on the whole to the integral for the force. Consider, for example, two terms in dw/dz of the form $A/(z-a)^n$, $B/(z-b)^m$ where n and m are positive integers and A, B, a, b are constants. The contribution of these two terms to $\oint (dw/dz)^2\, dz$ is

$$\oint \left[\frac{A^2}{(z-a)^{2n}} + \frac{2AB}{(z-a)^n\,(z-b)^m} + \frac{B^2}{(z-b)^{2m}} \right] dz.$$

Even the middle term here integrates to zero. For, the path of integration may be displaced toward infinity without crossing any singularity of the integrand and hence without changing the value of the integral; and toward infinity, using the binominal theorem,

$$\frac{1}{(z-a)^n\,(z-b)^m} = \left(\frac{1}{z^n} + \frac{na}{z^{n+1}} \ldots \right) \left(\frac{1}{z^m} + \frac{mb}{z^{m+1}} \ldots \right) = \frac{1}{z^{n+m}} + \ldots\ldots$$

This integrates to zero, since $n + m > 1$.

In the same way it may be seen that the same product term contributes nothing to the integral for the moment L provided $n + m > 2$.

(See Reference 1, Article 72b, where U, V replace $-U\cos\gamma$, $-V\sin\gamma$, and α, β replace b_1', b_1''; Reference 2, Section 6.41.)

75. THE LAGALLY THEOREMS

The following special case of the force action on a cylinder is readily handled by means of the Blasius theorem.

Let a uniform line source be located at $z = a$, outside a cylinder of any shape, and let the flow of the fluid be uniform at infinity, with velocity components $u = -U \sin \gamma$ and $v = -U \cos \gamma$. Then, if X and Y denote the x and y components of the force on unit length of the cylinder, it will be shown that

$$X = -\rho \Gamma U \sin \gamma + 2\pi\rho A\, u_{ac}, \quad Y = \rho \Gamma U \cos \gamma + 2\pi\rho A\, v_{ac} \qquad [75a, b]$$

Here Γ is the circulation around the cylinder; $2\pi A$ represents the volume of fluid emitted per second per unit length from the line source; and u_{ac}, v_{ac} are the components of the partial particle velocity at the location of the source caused by the presence of the cylinder, in addition to the velocity that would exist there if the cylinder and all circulation around it were removed and replaced by fluid.

To prove this theorem, the contour of integration in Equation [74g] is displaced from the contour C of the cylinder and transformed into a distant contour S together with a small contour σ surrounding $z = a$; see Figure 113 and compare Section 29. The value of the integral remains unaffected, since no singularities are passed in transforming the contour. The theorem of residues is then used, as explained in Section 30.

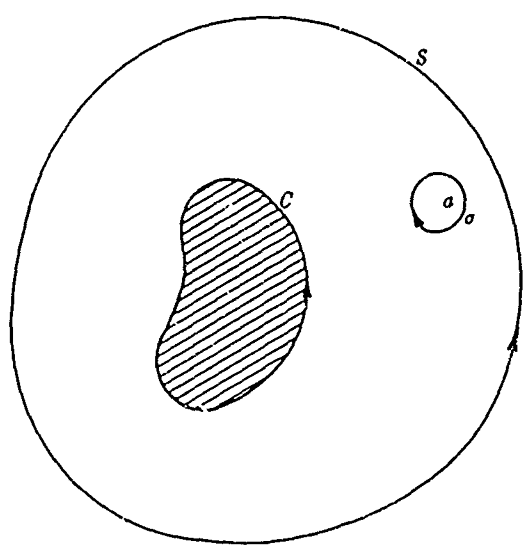


Figure 113 – A line source at "a" near a cylinder C

The complex potential can be written, from Equations [35a] and [40a],

$$w = -A \ln (z-a) + Uze^{-i\gamma} + w_c$$

where w_c is the partial potential due to the presence of the cylinder. Then

$$\frac{dw}{dz} = -\frac{A}{z-a} + Ue^{-i\gamma} + \frac{dw_c}{dz}, \quad -u_{ac} + iv_{ac} = \frac{dw_c}{dz}$$

from Equation [25i].

On S, $1/(z-a)$ can be expanded by the binomial theorem:

$$-\frac{A}{z-a} = -A\left(\frac{1}{z} + \frac{a}{z^2} + \frac{a^2}{z^3} + \ldots\ldots\right)$$

Furthermore, dw_c/dz will vanish at infinity, so that, on the contour S, w_c can be expanded as in Equation [72d]:

$$w_c = \frac{i\Gamma}{2\pi} \ln z + \frac{b_1}{z} + \frac{b_2}{z^2} \ldots\ldots, \quad \frac{dw_c}{dz} = \frac{i\Gamma}{2\pi z} - \frac{b_1}{z^2} - \frac{2b_2}{z^3} \ldots\ldots$$

All but two of the resulting terms in the integral around S then give zero.

On σ, which is traversed in the negative direction, the residue of $[-2A/(z-a)]$ $[dw_c/dz]$ contributes to the integral $4\pi i A\,[dw_c/dz]_{z=a} = 4\pi i A\,(-u_{ac} + i\,v_{ac})$, since dw_c/dz is analytic at $z=a$; see Section 30. The term in AU cancels one obtained from S.

Thus Equation [74g] becomes

$$X - iY = -\rho\Gamma U e^{-i\gamma} + 2\pi\rho A\,(u_{ac} - i\,v_{ac}). \qquad [75c]$$

As was seen in Section 40, A is real for a source; hence Equations [75a, b] follow.

If there is a *line vortex* instead of a source at $z = a$, with circulation Γ_0 around it, by [40m] $A = -i\,\Gamma_0/2\pi$; hence

$$X = -\rho\Gamma U \sin\gamma - \rho\Gamma_0 v_{ac}, \quad Y = \rho\Gamma U \cos\gamma + \rho\Gamma_0 u_{ac}. \qquad [75d, e]$$

If $U = 0$, a little reflection shows that the force on the cylinder is directed more or less towards the source or vortex.

The torque L on the cylinder per unit of its length is given by Equation [74h]. In evaluating this integral around σ, the factor z in the integrand can be written as $(z-a) + a$. It is found that

$$L = -\frac{1}{2}\rho(R)\left\{4\pi i a A(-u_{ac} + i v_{ac}) + 2\pi i \left(-\frac{i\Gamma A}{\pi} - \frac{\Gamma^2}{2\pi} - 2b_1 U e^{-i\gamma}\right)\right\} .$$

Write $b_1 = |b_1| e^{i\beta_1}$. Then, for a line source at $z = a$,

$$L = \rho\,[A\,(2\pi a v_{ac} - \Gamma) - 2\pi U |b_1| \sin(\beta_1 - \gamma)], \tag{75f}$$

or for a line vortex,

$$L = \rho\,[a\Gamma_0 u_{ac} - 2\pi U |b_1| \sin(\beta_1 - \gamma)]. \tag{75g}$$

To use these results, u_{ac} and v_{ac} must be known.

If several sources or vortices are present, the values of X, Y, and L due to all of them are simply added to obtain the total force and torque, as is easily verified. The principle of the superposition of flows will not hold, however, since in each formula the values of u_{ac} and v_{ac} are influenced indirectly by all sources or vortices. The sources or vortices may be fictitious, introduced to represent the effect of another cylinder, with or without circulation around it; in this way it may be possible to calculate the interaction of two or more cylinders.

(See Reference 2, Section 8.63, 9.53, 13.62; also Section 8.83 for an extension to dipoles.)

76. KINETIC ENERGY IN TRANSLATIONAL MOTION

By using the same stratagem as in Section 74, some useful formulas can be derived for the kinetic energy of the fluid surrounding a cylinder that is moving in translation perpendicularly to its generators. Let the velocity of the cylinder be U with the fluid at rest at infinity and with no circulation around the cylinder, and first, for simplicity, let it be moving toward positive x. Then the common normal velocity of the surface of the cylinder and the fluid, at a point where the x direction cosine of the normal to the surface is l, is q_n where $q_n = lU$, and the kinetic energy of the fluid per unit length of the cylinder is T_1 where, from Equation [17d],

$$T_1 = \frac{1}{2}\rho \oint \phi\, q_n\, ds = \frac{1}{2}\rho U \oint \phi\, l\, ds = \frac{1}{2}\rho U \oint \phi\, dy,$$

since $l\, ds = dy$. Here ρ is the density of the fluid and the integration is to be extended around the surface of the cylinder.

Now, if

$$w = \phi + i\psi \text{ and } z = x + iy,$$

then

$$\phi\, dy = (I')\,(w\, dz) - \psi\, dx,$$

where the symbol (I') signifies that only the imaginary part of what follows is to be taken, and without including the factor i; thus

$$(I')\,(w\, dz) = (I')\,[(\phi + i\psi)\,(dx + i\,dy)] = \phi\, dy + \psi\, dx.$$

Also, between two points on the surface ds apart, ψ differs by $d\psi$ where $d\psi = -\,q_n\, ds = -\,lU ds = -\,U\, dy$; whence, after integrating, $\psi = -\,Uy + C$ on the surface, where C is a constant, Substituting,

$$T_1 = \frac{1}{2}\rho U[(I')\,(\oint w\, dz) - U S] \qquad [76a]$$

where S stands for $-\oint y\, dx$ and represents the cross-section area of the cylinder. Here C has disappeared because $\oint C\, dx = C \oint dx = 0$.

This result may now be generalized so as to allow the cylinder to be moving at speed U in a direction inclined at an angle γ to the positive x-axis. Both the cylinder and the flow are rotated through an angle γ about the origin if z is replaced by $ze^{-i\gamma}$, as explained in Sections 25 and 34. Thus Equation [76a] is replaced by

$$T_1 = \frac{1}{2}\rho U\,[(I')\,(\oint w\, e^{-i\gamma}\, dz) - U S], \qquad [76b]$$

since the rotation does not alter S.

Another useful form for T_1 may be obtained by displacing the contour of integration toward infinity, which does not alter the value of the integral; see Section 29. Then w can be expanded as in Equation [72d], but here the last two terms of that expansion vanish, so that

$$w = a_0 + \frac{b_1}{z} + \frac{b_2}{z^2} \; \ldots\ldots \qquad [76c]$$

In the integral, all terms give zero except that $\oint b_1\, dz/z = 2\pi i b_1$; see Se ... Also, $(I')\,(i\, b_1\, e^{-i\gamma}) = (R)\,[b_1\, e^{-i\gamma}]$ where (R) indicates as in Section 74 that only the real part of what follows is to be taken. Hence

$$T_1 = \frac{1}{2}\,\rho\, U^2 S_1 \cdot \; S_1 = 2\pi\,(R)\left[\frac{b_1\, e^{-i\gamma}}{U}\right] - S, \qquad [76d\,e]$$

or, if the cylinder is moving toward positive x,

$$S_1 = 2\pi\,\frac{(R)\, b_1}{U} - S. \qquad [76f]$$

Finally, let the shape of the cylinder be changed by means of a transformation which is single valued and regular outside of the cylinder and which leaves the plane of z unchanged at infinity. Toward infinity such a transformation from z to a new variable z' can be written

$$z = z' + \frac{b_1'}{z'} + \frac{b_2'}{z'^2} \ldots \qquad [76g]$$

If this series is substituted for z in Equation [76c] as it stands, however, the boundary condition on the flow may no longer be satisfied at the surface of the cylinder. To avoid this difficulty, let the moving cylinder first be brought to rest by superposing uniform motion in the opposite direction; then, from Equation [35a], w becomes, in place of Equation [76c],

$$w_1 = U\, e^{-i\gamma}\, z + a_0 + \frac{b_1}{z} \ldots = U\, e^{-i\gamma}\, z' + a_0 + (U\, b_1' e^{-i\gamma} + b_1)\,\frac{1}{z'} \; \ldots\ldots$$

The boundary condition on the cylinder is now, on the z-plane, ψ = constant, and this condition remains satisfied on the z' plane. Removal of the term $U\, e^{-i\gamma}\, z'$ then sets the cylinder moving again, at speed U in a direction inclined at the angle γ to the positive real axis of z'; the corresponding flow is represented by

$$w = a_0 + (U\, b_1' e^{-i\gamma} + b_1)\,\frac{1}{z'} \; \ldots\ldots \qquad [76h]$$

Equation [76e] for the entrained area S_1' then gives

$$S_1' = 2\pi\,(R)\left(b_1' e^{-2i\gamma} + \frac{b_1\, e^{-i\gamma}}{U}\right) - S' \qquad [76i]$$

where S' is the cross-sectional area of the transformed cylinder.

(See Reference 74, where a proof not employing complex variables is given by Leathem.)

AIRFOILS

77. THE JOUKOWSKI TRANSFORMATION

By making a transformation from z to a new variable z', the flow with circulation around a circular cylinder can be transformed into the flow around a cylinder of a different shape. The equipotential curves and streamlines on the z-plane transform into curves on the z'-plane in association with the same values of ϕ and ψ. Since the total change in ϕ on going around a closed curve thus remains unaltered, the circulation around the transformed cylinder is the same as that around the original circular cylinder.

It was shown by Joukowski that a first step toward obtaining in this way useful profiles for airfoils could be taken by using the simple transformation

$$z' = z + \frac{c^2}{z}, \qquad [77a]$$

where c is a real positive constant, $z' = x' + iy'$, $z = x + iy$.
If c is chosen equal to the radius a of the given cylinder, z' becomes simply the complex potential w for the flow past the circular cylinder itself, as obtained in Section 69. Assume, therefore, that $c < a$.

The transformation Equation [77a] has a singularity at $z = 0$. Furthermore, since

$$\frac{dz'}{dz} = 1 - \frac{c^2}{z^2} = \frac{1}{z^2}\,(z + c)\,(z - c), \qquad [77b]$$

the transformation is not conformal at either of the points $z = \pm c$, or $z' = \pm 2c$, where $dz'/dz = 0$. In fact, as z passes through either of these points along a smooth curve, since dz'/dz changes sign, the motion of z' reverses, so that the corresponding z' curve exhibits a *cusp*.

Let the initial circular cylinder be so placed on the z-plane that neither of the points $z = \pm c$ lies outside of it. If one of them lies on the cylinder, an infinite velocity will occur at the corresponding point in the transformed flow unless the original point on the initial cylinder was a stagnation point. For $dw/dz' = (dw/dz)/(dz'/dz)$, so that, at $z = \pm c$, $dw/dz' \to \infty$ unless $dw/dz \to 0$. The singular point $z = 0$ lies inside the cylinder and can be disregarded.

To study the general character of the transformation, let z' and z both be represented for the moment on the same plane, with coincident axes. At infinity $z' = z$, so that the flow is unaltered. The transformation gives to every finite point represented by $z = re^{i\theta}$ the displacement $c^2/z = (c^2/r)e^{-i\theta}$; the magnitude of this displacement is inversely proportional to r and its direction lies at the same angle below the x-axis as does the vector representing z above it; see Figure 114. Thus all points not on the x-axis are moved toward this axis, and all points not on the y-axis are moved away from this axis, provided $r \geq c$. Points on the x-axis are merely shifted along it, and similarly for the y-axis.

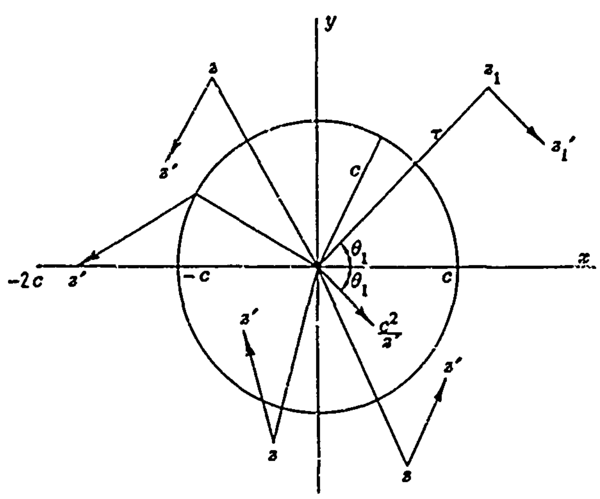


Figure 114 – The Joukowski transformation, $z' = z + c^2/z$.

Points lying on the circle $|z| = c$ are brought on to the segment of the x-axis between $z' = \pm 2c$. Other circles transform into curves whose shapes vary widely. A circle centered on the x-axis is transformed into a curve that is symmetric with respect to the x'-axis. If the center of the circle is not on either axis, the transformed curve is asymmetric.

The part of the z-plane that lies outside the circle $|z| = c$ is thus mapped onto the entire z'-plane, conformally except at $z = \pm c$. The transformation can be visualized by imagining the circle $|z| = c$ to be both flattened vertically and drawn out horizontally until it becomes a segment of the real axis of length $4c$, accompanied by a corresponding distortion of all parts of the plane. Circles centered at the origin become converted into confocal ellipses, while the radial lines outside the c-circle become joined at the ends to form hyperbolas confocal with the ellipses; see Section 81. The interior of the c-circle is likewise mapped onto the entire z'-plane, as if it were turned inside out and also reflected in the real axis while the origin recedes to infinity.

The transformed cylinder. Let the center of the circle of radius a, which represents a circular cylinder, be located on the z-plane at the point

$$z = z_1 = h\,e^{i\eta} \qquad [77c]$$

where h and η are real positive constants, or at the point ($h \cos \eta$, $h \sin \eta$); and let the fluid at infinity have a velocity U inclined at an angle γ to the negative x-axis, with components $-U \cos \gamma$, $-U \sin \gamma$; see Figure 115. Then, from Equation [69j], the complex potential is

$$w = U \left[(z - h\,e^{i\eta})\, e^{-i\gamma} + \frac{a^2}{z - h\,e^{i\eta}}\, e^{i\gamma} \right] + \frac{i\Gamma}{2\pi} \ln \frac{z - h\,e^{i\eta}}{a} \qquad [77d]$$

By substitution for z from Equation [77a], w can be found, if necessary, as a function of z'.

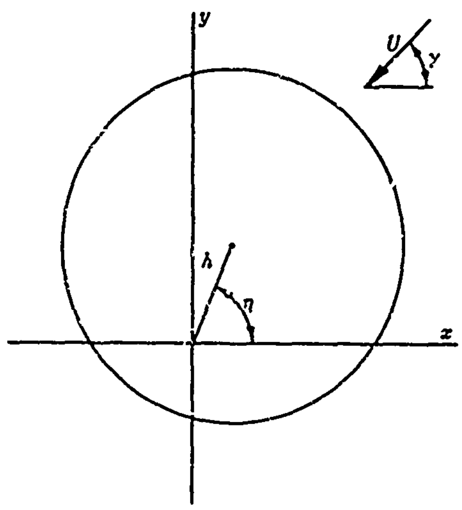


Figure 115 – Diagram for Equation [77d].

The lift on either cylinder is given by the Kutta-Joukowski formula [73a]. A simple formula for the torque on the transformed cylinder can be obtained from the Blasius theorem. For this purpose, dw/dz' must be expanded in descending powers of z', as in Equation [74j], but terms in higher negative powers than $1/z'^2$ are not needed here. From Equations [77d] and [77b]

$$\frac{dw}{dz'} = \frac{dw}{dz} \bigg/ \frac{dz'}{dz} = \left(1 - \frac{c^2}{z^2}\right)^{-1} \left[U\,e^{-i\gamma} - \frac{a^2\,U}{(z - h\,e^{i\eta})^2}\, e^{i\gamma} + \frac{i\Gamma}{2\pi}\, \frac{1}{z - h\,e^{i\eta}} \right] \qquad [77e]$$

Solving Equation [77a] as a quadratic in z and expanding the radical,

$$z = z'\left[\frac{1}{2} + \frac{1}{2}\left(1 - \frac{4c^2}{z'^2}\right)^{1/2}\right] = z' - \frac{c^2}{z'} - \frac{c^4}{z'^3} \;\ldots\ldots,$$

$$\frac{1}{z} = \frac{1}{z'}\left[1 - \left(\frac{c^2}{z'^2} + \frac{c^4}{z'^4} \ldots\right)\right]^{-1} = \frac{1}{z'} + \frac{c^2}{z'^3} + \ldots\ldots$$

by the series, $(1-x)^{-1} = 1 + x + \ldots\ldots\ldots$; hence

$$(z - h\,e^{i\eta})^{-1} = \frac{1}{z} + \frac{h\,e^{i\eta}}{z^2} \ldots\ldots = \frac{1}{z'} + \frac{h\,e^{i\eta}}{z'^2} + \ldots\ldots,$$

$$(z - h\,e^{i\eta})^{-2} = \frac{1}{z^2} + \ldots\ldots = \frac{1}{z'^2} + \ldots\ldots; \left(1 - \frac{c^2}{z^2}\right)^{-1} = 1 + \frac{c^2}{z^2} + \ldots\ldots = 1 + \frac{c^2}{z'^2} + \ldots\ldots$$

Hence, as far as terms of order $1/z'^2$,

$$\frac{dw}{dz'} = U\,e^{-i\gamma} + \frac{i\Gamma}{2\pi}\,\frac{1}{z'} + \left[\frac{i\Gamma}{2\pi}\,h\,e^{i\eta} + U\,(c^2\,e^{-i\gamma} - a^2\,e^{i\gamma})\right]\frac{1}{z'^2} \;\ldots\ldots$$

Thus the constant b_1 in Equation [74j] with z replaced by z' here has the value

$$b_1 = -\frac{i\Gamma}{2\pi}\,h\,e^{i\eta} - U\,(c^2\,e^{-i\gamma} - a^2\,e^{i\gamma}), \qquad [77f]$$

and, upon substituting in Equation [74m] and selecting the real part as indicated by the symbol (R), the torque per unit length on the cylinder about an axis passing through the origin of coordinates is found to be

$$N = -2\pi\rho\,c^2\,U^2 \sin 2\gamma + \rho\,h\,\Gamma\,U \cos(\eta - \gamma). \qquad [77g]$$

(See Reference 2, Section 7.50, where the sign of γ is reversed; Reference 4.)

78. CIRCULAR ARCS BY THE JOUKOWSKI TRANSFORMATION

Let the initial circle representing a circular cylinder be centered now on the y-axis at $(0, h)$, and let it have such a radius a as to pass through the points $z = \pm c$, as in Figure 116. Then it transforms into a circular arc with ends at the points $z' = \pm 2c$, which may represent a lamina of arcuate cross section.

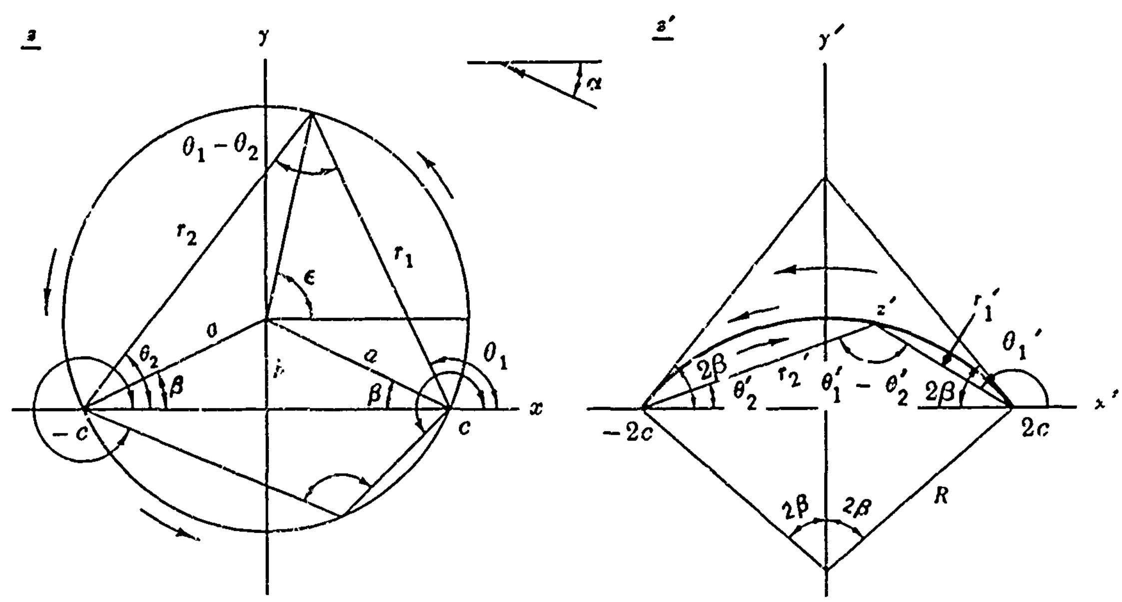


Figure 116 – The z-circle goes into an arc on the z' plane. See Section 78.

For, each of the following two equations is equivalent to Equation [77a]

$$(z' - 2c)\, z = (z - c)^2, \quad (z' + 2c)\, z = (z + c)^2.$$

Division of these equations gives

$$\frac{z' - 2c}{z' + 2c} = \left(\frac{z - c}{z + c}\right)^2. \qquad [78a]$$

Write

$$z - c = r_1 e^{i\theta_1}, \quad z + c = r_2 e^{i\theta_2}, \quad z' - 2c = r_1' e^{i\theta_1'}, \quad z' + 2c = r_2' e^{i\theta_2'}.$$

The angles thus introduced are illustrated in Figure 116. Then, equating complex amplitudes on both sides of Equation [78a],

$$\theta_1' - \theta_2' = 2\,(\theta_1 - \theta_2). \qquad [78b]$$

Now, it is clear from Equation [78a] that the points $z = \pm c$ correspond to $z' = \pm 2c$. As z, starting from $z = c$, traverses the initial circle positively up to $z = -c$, $\theta_1 - \theta_2$ retains a constant value, for a geometrical reason; hence by Equation [78b] $\theta_1' - \theta_2'$ likewise remains constant, and z', therefore, traces a circular arc extending from $z' = 2c$ to $z' = -2c$. As z continues past $z = -c$, θ_2 changes by π, and $\theta_1' - \theta_2'$ by 2π; this is easily seen to be equivalent to no change at all in $\theta_1' - \theta_2'$, so that z' must now retrace the arc, arriving back at $z' = 2c$ as z comes to c.

The constant value of $\theta_1 - \theta_2$ along the upper part of the circle can be written

$$\theta_1 - \theta_2 = \frac{\pi}{2} - \beta \qquad [78c]$$

where the significance of β is shown in Figure 116a, and

$$c = a \cos \beta, \; h = a \sin \beta = c \tan \beta. \qquad [78d, e]$$

The angle between the tangent and the chord at each end of the arc on the z'-plane is $\pi - (\theta_1' - \theta_2')$ or 2β; thus the arc has a total angular length of 4β. Its radius R, and its camber C, or the ratio of its maximum height above the chord to the length of the chord, are, from the geometry of Figure 116a and Equation [78d, e],

$$R = \frac{2c}{\sin 2\beta} = \frac{a^2}{\sqrt{a^2 - c^2}}, \; C = \frac{R\,(1 - \cos 2\beta)}{2R \sin 2\beta} = \frac{1}{2} \tan \beta. \qquad [78f, g]$$

The interior and the exterior of the circle are each mapped onto the entire z'-plane; the mapping of the interior is to be ignored here.

For an application, it will be convenient to assume the fluid to approach at infinity from a direction inclined at an angle α *below* the positive x- or x'-axis, with components of velocity $-U \cos \alpha$, $+U \sin \alpha$. Since the flow at infinity remains unaltered, the angle of approach is the same in the transformed as in the original flow. Then, in Equation [77d] for the complex potential w, $\gamma = -\alpha$; and here $\eta = \pi/2$, $e^{i\eta} = i$. Thus

$$w = U \left[(z - ih)\, e^{i\alpha} + \frac{a^2}{z - ih}\, e^{-i\alpha} \right] + \frac{i\Gamma}{2\pi} \ln \frac{z - ih}{a}. \qquad [78h]$$

Substitution for z in terms of z' from Equation [77a] then yields w as a function of z', and from $w = \phi + i\psi$, $z' = x' + iy'$, the potential ϕ and stream function ψ can be found; but the equations are complicated. The flow net is most easily constructed by direct graphical transformation of that for the cylinder; a sufficient number of points on the curves can be transferred into their new positions by the method described in Section 77 and illustrated in Figure 114.

From Equations [78h] and [77a]

$$-u'+iv' = \frac{dw}{dz'} = \frac{dw}{dz}\Big/\frac{dz'}{dz} = \frac{z^2}{z^2-c^2}\left[U\left(e^{i\alpha} - \frac{a^2}{(z-ih)^2}\,e^{-i\alpha}\right) + \frac{i\Gamma}{2\pi(z-ih)}\right] \qquad [78i]$$

from which the components of velocity u' and v' in the transformed flow can be found.

On the arc itself, since the corresponding z-point lies on the initial circle, z is represented by $z = ih + a e^{i\epsilon}$ where ϵ is a variable angle, shown in Figure 116. Hence

$$e^{i\alpha} - \frac{a^2}{(z-ih)^2}\,e^{-i\alpha} = e^{-i\epsilon}\left(e^{i(\alpha+\epsilon)} - e^{-i(\alpha+\epsilon)}\right) = 2ie^{-i\epsilon}\sin(\alpha+\epsilon),\quad \frac{i\Gamma}{2\pi(z-ih)} = i\,\frac{\Gamma}{2\pi a}\,e^{-i\epsilon},$$

and, since $z^2 - c^2 = (z+c)(z-c)$ and $|ie^{-i\epsilon}| = 1$,

$$q = \left|\frac{dw}{dz'}\right| = \left|\frac{r^2}{r_c\,r_{-c}}\right|\left|\left(2U\sin(\alpha+\epsilon) + \frac{\Gamma}{2\pi a}\right)\right| . \qquad [78j]$$

Here $r = |z|$, $r_c = |z-c|$, $r_c = |z+c|$; and these quantities represent distances that can be measured on a plot. The point on the arc at which q as calculated from Equation [78j] is the velocity can be found by graphical transfer of the corresponding point on the circle.

Streamlines for $\beta = 11$ deg, $\alpha = 25$ deg and $\Gamma = 0$ are shown in Figure 117; the diagram has been tipped up to save space. Another case of streamlines about a flattish arc is shown in Figure 118; see Reference 114. About a semicircular arc, streamlines for three cases of non-circulatory flow are shown in Figure 119; see Reference 113.

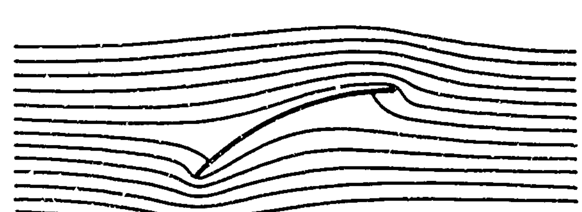


Figure 117 – Streamlines past a circular arc without circulation.

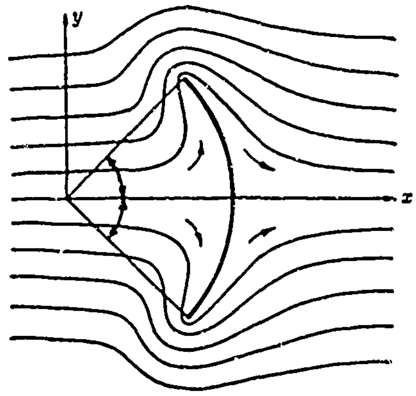


Figure 118 – Streamlines past a circular arc with no circulation about it.

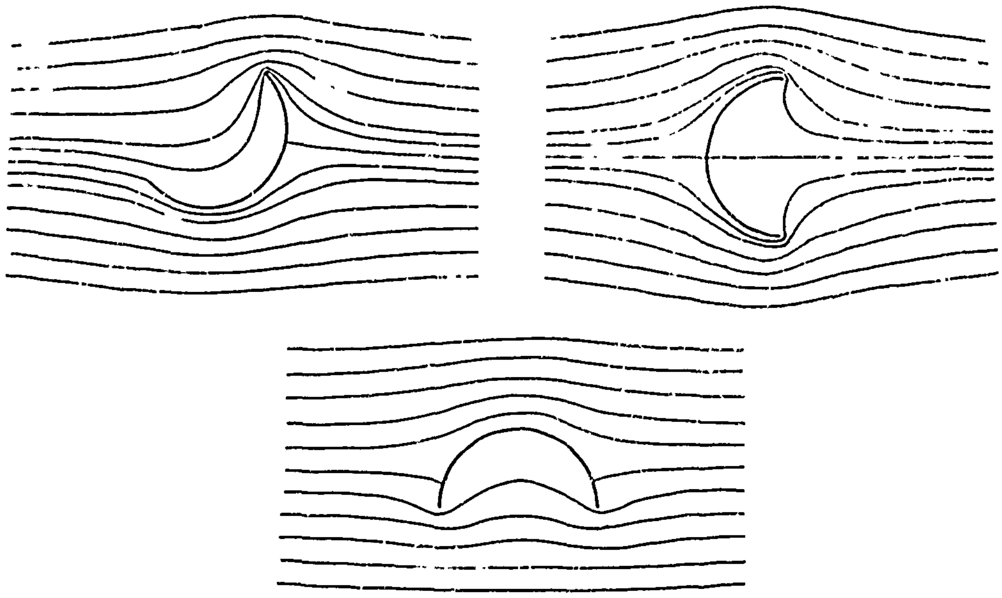


Figure 119 – Noncirculatory streamlines about a semicircular arc. (Copied from Reference 113)

The lift per unit length on the lamina, in the direction perpendicular to the direction of the steady stream at infinity, is in any case $L = \rho \Gamma U$, by the Kutta-Joukowski theorem. The torque about the origin of coordinates or the center of the chord, from Equation [77g], in which here $\gamma = -\alpha$, $\eta = 90$ deg, $h = c \tan \beta$, is, in any case,

$$N = 2\pi \rho c^2 U^2 \sin 2\alpha - \rho c \Gamma U \tan \beta \sin \alpha \qquad [78k]$$

Here ρ is the density of the fluid.

The lift and torque will be the same if the lamina is itself moving through fluid which is at rest at infinity; the lift is then perpendicular to the direction of motion of the lamina, and α is the angle between the direction of motion and the chord of the lamina, as shown in Figure 120.

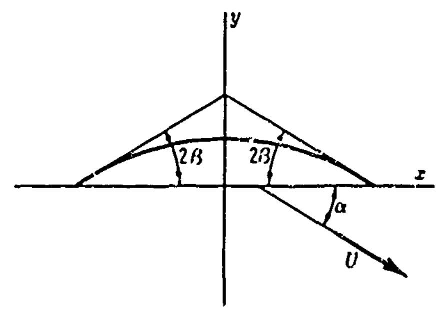


Figure 120 – Symbol relations for oblique motion of a circular-arc lamina.

At the edges of the lamina, where $z = \pm c$, $q \to \infty$, in general. The velocity can be made finite at one edge, however, by choosing the ratio Γ/U so that the bracket in Equation [78i] vanishes at that edge. Then $dw/dz = 0$ there and the corresponding point on the initial cylinder is a stagnation point. In particular, let

$$U\left(e^{i\alpha} - \frac{a^2}{(-c-ih)^2}\,e^{-i\alpha}\right) = -\frac{i\Gamma}{2\pi(-c-ih)}, \qquad [78l]$$

or, since by Equation [78d, e]

$$c + ih = a(\cos\beta + i\sin\beta) = ae^{i\beta}, \qquad [78m]$$

$$\Gamma = 4\pi a U \sin(\alpha + \beta). \qquad [78n]$$

Then, eliminating Γ between Equations [78i] and [78l] and at the very end using Equation [78m],

$$\frac{dw}{dz'} = \frac{Uz^2}{(z-c)(z-ih)}\left(e^{i\alpha} - \frac{a}{z-ih}\,e^{-i(\alpha+\beta)}\right).$$

Thus dw/dz' and q are now finite at $z = -c$, although still infinite at $z = c$. Furthermore, both dw/dz' and z' are continuous functions of z at $z = -c$, so that no discontinuity of the velocity can occur there; the fluid flows smoothly away from the trailing edge along the tangent.

With this value of Γ, Equation [78j] becomes

$$q = \frac{2r^2U}{r_c\, r_{-c}}\left[\sin(\alpha + \epsilon) + \sin(\alpha + \beta)\right], \qquad [78o]$$

and the lift is

$$L = \rho \Gamma U = 4 \pi \rho a U^2 \sin(\alpha + \beta), \qquad [78p]$$

where $a = c/\cos\beta$. The lift in this case is directed away from the convex side of the lamina provided $\alpha > -\beta$, and it is a maximum for $\alpha = 90 \text{ deg} - \beta$.

The flow net around such an arc-shaped lamina, with the circulation adjusted to make the velocity finite at the trailing edge, is shown in Figure 121. The stream approaches from the left at an angle of 10 deg to the chord. Because of the presence of circulation, the apparent directions of approach and departure differ in the figure by a few degrees. The theoretical pressure-differences on both surfaces of the lamina are plotted in Figure 122, drawn vertically from the arc as a base; the numbers represent millimeters of water in an airstream of 10 meters per second. The broken line represents old measurements by Eiffel.

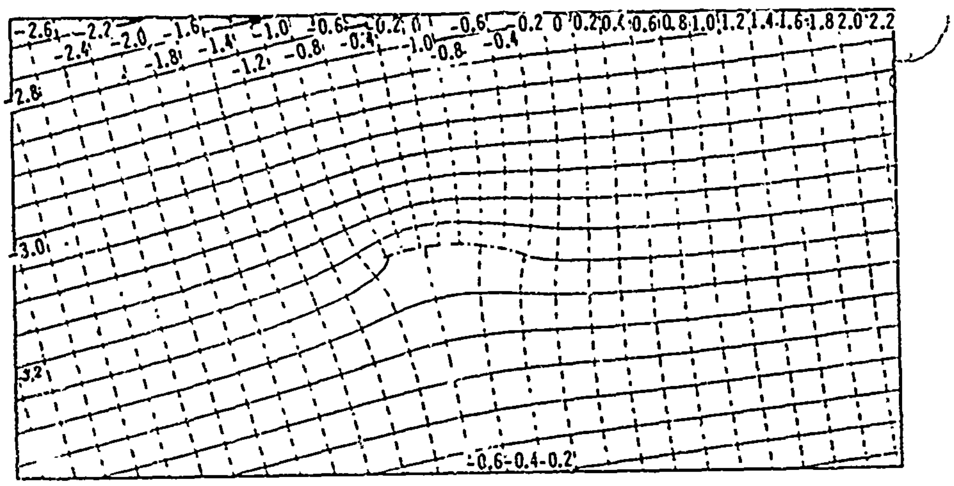


Figure 121 – Flow with finite trailing velocity around an arc-shaped lamina.

The *kinetic energy* of the fluid when the lamina moves in translation is easily found. Let it move at velocity U at the angle α with the direction of its chord, with no circulation around it, and with the fluid at rest at infinity. From Equation [78h], in which the term in $e^{i\alpha}$ represents the uniform stream and is to be dropped, the appropriate potential is

$$w = \frac{a^2 U}{z - ih} e^{-i\alpha} = \frac{a^2 U e^{-i\alpha}}{z} + \frac{ih\, a^2 U}{z^2} e^{-i\alpha} \dots\dots,$$

with α measured downward from the positive real axis. The transformation Equation [77a] can be written in descending powers of z' as

$$z = z' - \frac{c^2}{z} = z' - \frac{c^2}{z'} + \ldots\ldots\ldots$$

Comparison of these equations with Equations [76d] and [76h] shows that here $b_1 = a^2 U e^{-i\alpha}$, $b_1' = -c^2$; also, here $\gamma = -\alpha$. Hence from Equation [76e, i], in which $S' = 0$, the kinetic energy of the fluid per unit length of the lamina is

$$T_1 = \frac{1}{2} \rho S_1' U^2 = \pi \rho U^2 (a^2 - c^2 \cos 2\alpha), \quad [78q]$$

since $(R) e^{2i\alpha} = \cos 2\alpha$. A more useful form is obtained by writing b for $2c$, the half-chord of the lamina, and introducing its central height d above the chord. Using Equations [78g] and [78d], $d^2 = C^2 (4c)^2 = b^2 \tan^2 \beta = b^2 (a^2/c^2 - 1)$, whence $a^2 = (b^2 + d^2)/4$. Thus

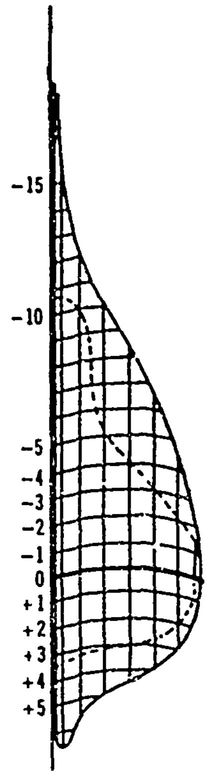


Figure 122 – Pressure differences above and below the lamina shown in Figure 121.

$$T_1 = \frac{1}{2} \rho \pi U^2 \left(b^2 \sin^2 \alpha + \frac{d^2}{2} \right). \quad [78r]$$

(See Reference 1, Article 70.)

79. THE JOUKOWSKI AIRFOILS

By displacing the initial circle so that it passes through only one of the points $z = \pm c$ and surrounds the other, the Joukowski transformation can be made to yield a contour that is pointed at one end and rounded at the other. If the circulation is then chosen so as to make the velocity finite at the pointed end, it is finite all round. According to a hypothesis proposed by Joukowski, a properly designed airfoil automatically develops in the fluid around it, by means of friction, a circulation of such magnitude as to remove the tendency for the velocity to become infinite at the trailing edge, which is usually made comparatively sharp, and measurements have shown this hypothesis to be close to the truth.

The general Joukowski transformation is most easily handled by a graphical method. It is desired to construct points representing z' where

$$z' = z + \frac{c^2}{z} \qquad [79a]$$

and c is real. If $z = re^{i\theta}$

$$\frac{c^2}{z} = \frac{c^2}{r} e^{-i\theta}.$$

Thus the point representing the complex number c^2/z lies on a line inclined at the angle θ below the x-axis, and at a distance c^2/r from the origin. The vectors representing c^2/z and z are easily constructed and can then be added vectorially to obtain z.

For values of z representing points on the initial circle, the operation can be simplified by first constructing the locus on which c^2/z must lie. Let the center C of the initial circle, whose radius is a, be displaced a distance h from the origin in a direction making an angle η with the positive x-axis, as illustrated in Figure 123. Then, when z lies on the circle,

$$(r \cos \theta - h \cos \eta)^2 + (r \sin \theta - h \sin \eta)^2 = a^2,$$

$$r^2 - 2\,hr\,(\cos \theta \cos \eta + \sin \theta \sin \eta) + h^2 - a^2 = 0.$$

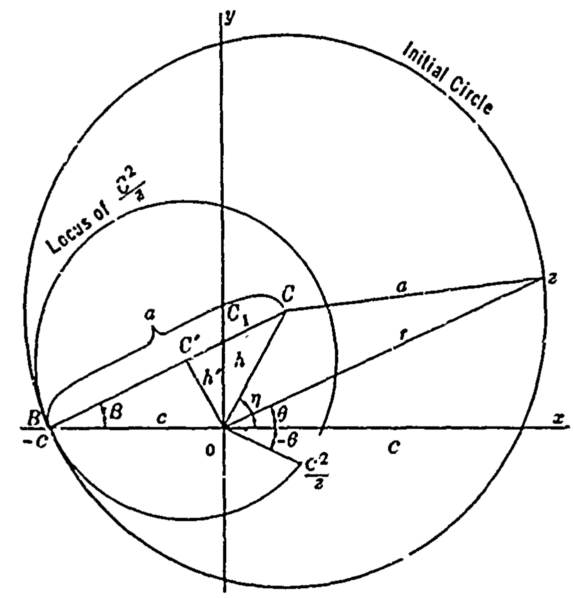


Figure 123 – Illustrating relations for a general Joukowski transformation.

Multiplying through by $-c^4/[r^2\,(a^2 - h^2)]$:

$$\left(\frac{c^2}{r}\right)^2 + 2\,\frac{c^2 h}{a^2 - h^2}\,\frac{c^2}{r}\,(\cos\theta\cos\eta + \sin\theta\sin\eta) - \frac{c^4}{a^2 - h^2} = 0.$$

This can be written

$$\left(\frac{c^2}{r}\cos\theta + h'\cos\eta\right)^2 + \left(-\frac{c^2}{r}\sin\theta - h'\sin\eta\right)^2 = a'^2, \qquad [79b]$$

$$h' = \frac{c^2 h}{a^2 - h^2}\,,\quad a' = \frac{c^2 a}{a^2 - h^2}\,. \qquad [79c, d]$$

Now $c^2\cos\theta/r$, $-c^2\sin\theta/r$ are the coordinates of the point c^2/z. Hence, Equation [79b] shows that c^2/z lies on a circle of radius a' drawn about the point C' or $(-h'\cos\eta,\ h'\sin\eta)$ as center. Clearly OC' and OC are equally inclined to the y-axis but on opposite sides of it.

If, in particular, the initial circle passes through the point $z = -c$ or B, C' lies on the radius BC. For, the slopes of BC and BC' are, respectively.

$$\frac{h\sin\eta}{c + h\cos\eta}\,,\quad \frac{h'\sin\eta}{c - h'\cos\eta} = \frac{c^2 h\sin\eta}{c\,(a^2 - h^2) - c^2 h\cos\eta}$$

from Equation [79c]. But, from the triangle BOC,

$$a^2 - h^2 = c^2 + 2\,c\,h\cos\eta.$$

Hence the second slope equals the first. Since the point $z = -c$ or B is itself on the locus circle, the two circles touch at B.

According to the results of Section 78, a circle centered at C_1, the intersection of the radius BC with the y-axis, would transform into a circular arc of total angular length 4β, where β is the angle between the radius BC and the x-axis. This arc, with ends at $z' = \pm 2c$, lies inside the transformed contour as a sort of skeleton.

The construction of an airfoil contour in this manner is shown in Figure 124 for $h = 0.87c$, $\beta = 34^\circ\,40'$. The skeleton arc is also drawn. The graphical procedure is discussed further by Ruden.[75]

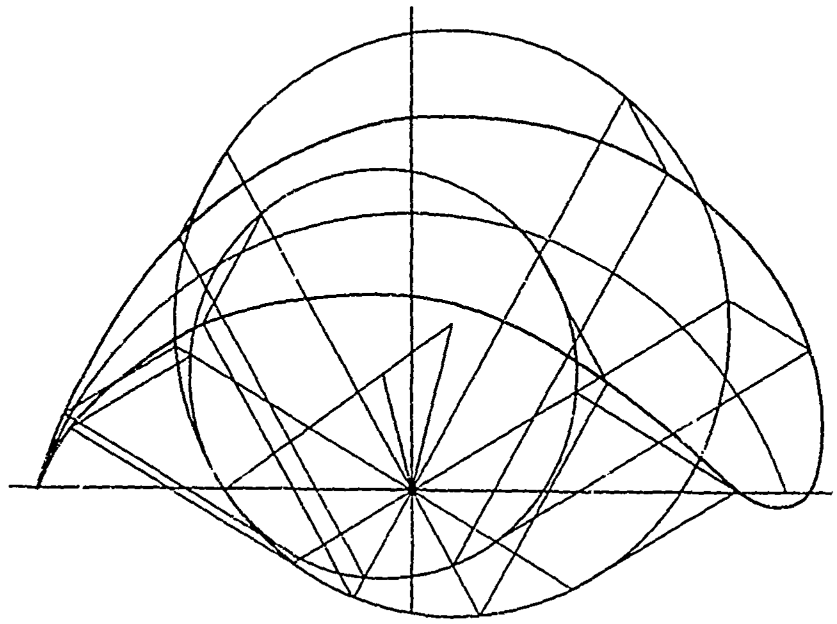


Figure 124 – Illustrating the construction of an airfoil contour.

If $\beta = 0$, so that the initial circle is centered on the x-axis, the skeleton arc becomes a straight line and the contour obtained from the circle is symmetrical about the x'-axis. Its shape depends on the ratio a/c. If $\beta \neq 0$, the contour is asymmetric.

For the flow in the surrounding fluid, nothing needs to be changed in the discussion of the last section except that here the velocity can be infinite only at the sharp edge or at $z = -c$, and ih is to be replaced by $h \cos \eta + ih \sin \eta = h e^{i\eta}$, as in Section 77. With the latter change, Equations [78i] and [78j] for u', v' and q, which are expressed in terms of quantities on the z-plane, hold as before.

If $\Gamma = 4 \pi a U \sin (\alpha + \beta)$, the velocity is again finite everywhere. Here U is the relative velocity of the airfoil and the fluid at infinity and α is the angle of attack, or the angle between the direction of approach of the fluid and the chord of the skeleton arc, taken positive when the approach is from the concave or less convex side. With this value of Γ, the lift per unit length is again

$$L = 4 \pi \rho a U^2 \sin (\alpha + \beta) \qquad [79e]$$

where ρ is the density of the fluid.

In any case $L = \rho \Gamma U$, provided the motion is steady, as for any cylinder.

The torque about the origin of coordinates on the z' plane is, from Equation [77g], in which here $\gamma = -\alpha$,

$$N = 2\pi\rho c^2 U^2 \sin 2\alpha + \rho h \Gamma U \cos(\eta + \alpha). \qquad [79f]$$

Flow without circulation past a Joukowski airfoil of a certain shape is shown in Figure 125. For another profile, there is shown in Figure 126 first, the flow without circulation, then the flow due to the circulation alone, and finally the resultant flow due to the superposition of the two; the circulation has been chosen so as to make the resultant velocity finite at the trailing edge.

The trailing edge can be rounded off by allowing the initial circle to enclose both of the singular points ($\pm c$, 0). A Joukowski profile constructed by R.H. Smith[251] with $\beta = 0$, $a/c = 1.35$, $h/c = 0.135$ is shown by the heavy curve with two rounded ends in Figure 127. Streamlines for the flow without circulation past this contour are shown in Figure 128.

The extended contour with a pointed end in Figure 127 represents the profile of U.S. Navy strut No. 2. It can be reproduced without visible error, using the method of Section 56, by assuming 5 line sources and 8 line sinks of suitable strength properly disposed along the axis. Figure 127 shows also the calculated distributions of pressure over both theoretical profiles in comparison with the observed distribution over strut No. 2 in an air stream; the flow is from right to left.

(See Reference 1, Article 70; Reference 2, Sections 7.20, 7.30, 7.31, 7.32, 7.40, 7.50.)

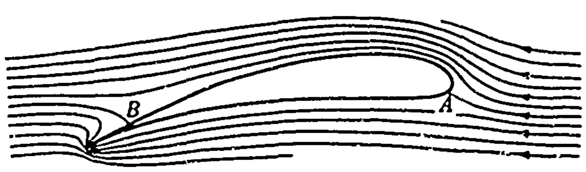


Figure 125 – Flow without circulation past a Joukowski airfoil. A and B indicate stagnation lines.

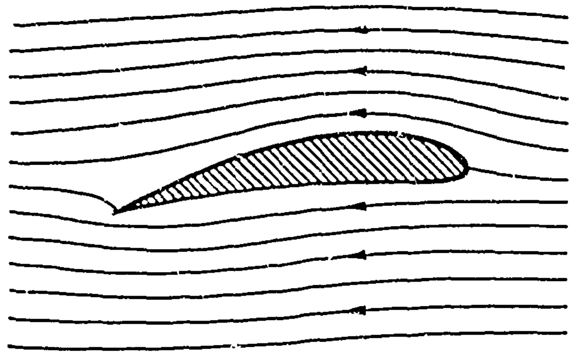


Figure 126a

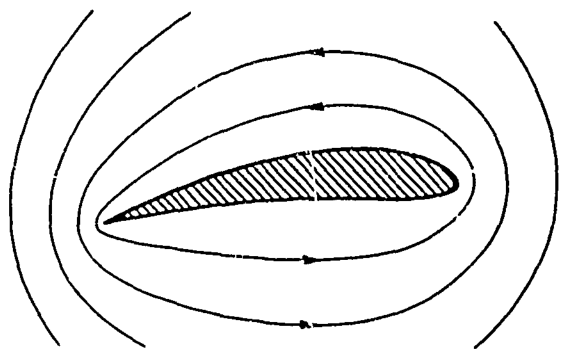


Figure 126b

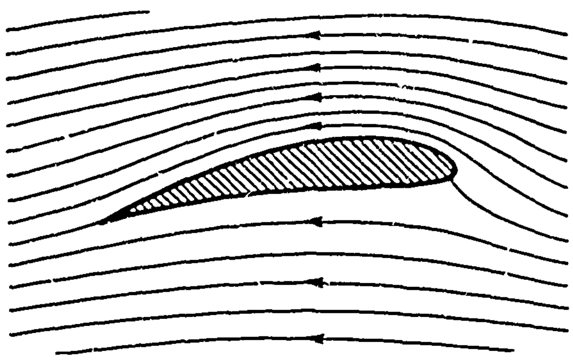


Figure 126c

Figure 126 – Streamlines past another Joukowski airfoil: (a) with no circulation, (b) with circulation only, (c) with circulation adjusted to eliminate cross flow at the trailing edge.

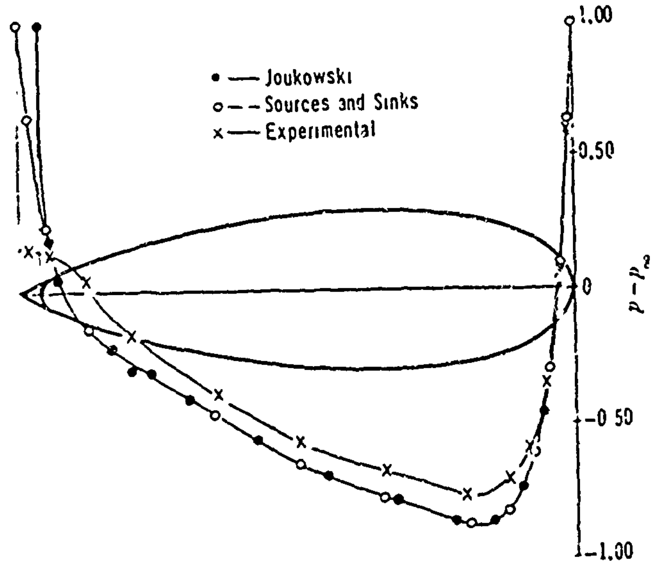


Figure 127 – Profile of U.S. Navy Strut No. 2 (pointed) and closely similar Joukowski profile (rounded), with theoretical and experimental pressure distributions. See the end of Sec. 79.

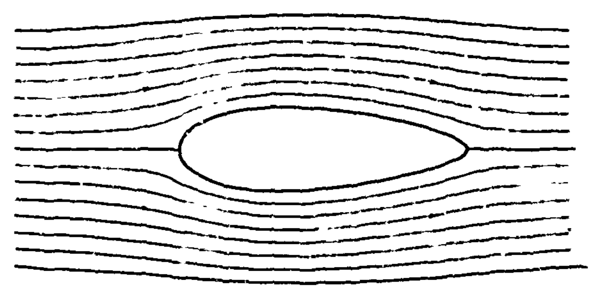


Figure 128 – Flow without circulation past a similar profile with rounded ends.

80. IMPROVED AIRFOILS

A modification of the Joukowski transformation by which the sides of the airfoil may be made to form a trailing edge containing a finite angle instead of a cusp is furnished by the circular-arc transformation, which will be treated in Section 88. There is no simple graphical construction for this transformation. It was studied by von Kármán and Trefftz[77], Müller[78], and Glauert[184], and was generalized further by Betz and Keune[79], who added a dipole term.

Other closed transformations for the construction of airfoil contours were discussed by Blasius[80], Piercy, Piper and Preston[81], Piper[82], and Durington and Dobbie[83]. More generally, any ordinary closed curve can be transformed into a circle by a suitable transformation, in order to make the two planes agree at infinity, the transformation must be representable by a Laurent series of the form mentioned in Section 76 or

$$z' = z + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots\cdots$$

The use of transformations so defined was studied by von Mises[84] and by Müller[85].

Simple methods for the treatment of thin airfoils were described by Jeffreys[86], Munk[87], Glauert[4, 88], and Millikan[89], who also considered biplanes.

Later work has been concerned chiefly with practical methods of making calculations for given profiles of any shape. See especially papers by Theodorsen and Garrick[90], and by Gebelein[91], also Theodorsen[92], Schmieden[93], Garrick[94], Kaplan[95]. Line sources on the axis of the airfoil are used by Pistolesi[96] and by Goldstein[97], both sources and vortices by Keune[98]. Jones and Cohen[99] show how to use the Joukowski transformation itself in order to effect small changes in a given profile.

Approximate methods for double or biplane airfoils have been discussed by Millikan[89] and, with use of elliptic functions, by Garrick[100].

The theoretical literature on airfoils is naturally extensive, but most of it either makes little use of potential theory or deals with systems of vortices and so lies outside of the scope of the present discussion.

VARIOUS CYLINDERS

81. CIRCLES INTO ELLIPSES

The transformation of Section 77,

$$z = z'' + \frac{c^2}{z''} \qquad [81a]$$

can be used to convert a circular cylinder into one of elliptic cross section.

For, consider a circle on the z''-plane centered at the origin, which can be described in terms of polar coordinates r, θ as follows:

$$z'' = x'' + iy'' = re^{i\theta}, \quad x'' = r\cos\theta, \quad y'' = r\sin\theta, \qquad [81b, c, d]$$

where r is constant. For the corresponding transformed curve on the z-plane, from Equation [81a],

$$z = x + iy = re^{i\theta} + \frac{c^2}{r}e^{-i\theta}, \qquad [81e]$$

$$x = \left(r + \frac{c^2}{r}\right)\cos\theta, \quad y = \left(r - \frac{c^2}{r}\right)\sin\theta. \qquad [81f, g]$$

The equation of the transformed curve can also be written

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \; a = r + \frac{c^2}{r}, \; b = \left| r - \frac{c^2}{r} \right|, \qquad [81h, i, j]$$

which shows that the given z''-circle becomes an ellipse on the z-plane having semiaxes a, b. Its foci are located at $y = 0$ and $x = \pm\sqrt{a^2 - b^2} = \pm 2c$.

Similarly, a radius from the origin of z'', on which θ is constant, becomes a curve on which, from Equation [81f, g]

$$\frac{x}{\cos\theta} + \frac{y}{\sin\theta} = 2r, \; \frac{x}{\cos\theta} - \frac{y}{\sin\theta} = \frac{2c^2}{r}, \; \frac{x^2}{4c^2\cos^2\theta} - \frac{y^2}{4c^2\sin^2\theta} = 1. \qquad [81k]$$

This represents a hyperbola having semiaxes $2c\cos\theta$, $2c\sin\theta$, and foci likewise at $(\pm 2c, 0)$. Since the transformation is conformal, the hyperbolas and ellipses are orthogonal, as were the original circles and radii.

Toward infinity, $z \to z''$ and the two planes become alike. The ellipses then reduce to circles like those on the z''-plane, and the hyperbolas approach their asymptotes, which have the directions of the original z'' radii.

In working with these curves, it is convenient to change somewhat the variables that characterize them. Let a new complex variable ζ be defined by

$$z'' = c\,e^{\zeta}, \; \zeta = \xi + i\eta. \qquad [81l, m]$$

Since $\zeta = \ln(z''/c)$ and $z'' = r\,e^{i\theta}$,

$$\xi = \ln(r/c), \; \eta = \theta. \qquad [81n, o]$$

Substitution in Equation [81a] then gives

$$z = 2c\cosh\zeta. \qquad [81p]$$

This latter transformation will be studied in the next section.

Each of the ellipses previously described now corresponds to a certain numerical value of ξ. There are two different circles on the z''-plane corresponding to each ellipse, however, one lying inside of the circle $r = c$ and the other outside of it; their radii r_1, r_2 are such that $r_1 r_2 = c^2$, since in Equation [81i, j] two such values of r give the same a and b, and Equation [81n] shows that for the larger circle $\xi > 0$ while for the smaller $\xi < 0$. Each hyperbola corresponds to $\eta = \eta_1 \pm 2n\pi$ where η_1 is a real number and n is an integer or zero.

The inverse transformation to Equation [81a] is

$$z'' = \frac{1}{2}\,[z \pm (z^2 - 4\,c^2)^{1/2}]; \tag{81q}$$

the plus sign goes with $r > c$ or $\xi > 0$, and the negative with $r < c$ or $\xi < 0$, as is easily verified for positive real z; for a real $z < -2\,c$, however, the symbol $(z^2 - 4\,c^2)^{1/2}$ must be understood to stand for the negative square root.

(See Reference 2, Section 6.30, where c is replaced by $c/2$, also Section 6.32.)

82. ELLIPTIC COORDINATES

Let

$$z = x + iy = c \cosh \zeta, \quad \zeta = \xi + i\,\eta. \tag{82a, b}$$

This transformation was studied briefly in a different notation in Section 61, and the results obtained there will be assumed. From Equations [61b, c, g, h, i, j]

$$x = c \cosh \xi \cos \eta, \; y = c \sinh \xi \sin \eta, \tag{82c, d}$$

$$\cosh \xi = \frac{1}{2c}\,\{[(x + c)^2 + y^2]^{1/2} + [(x - c)^2 + y^2]^{1/2}\}, \tag{82e}$$

$$\cos \eta = \frac{1}{2c}\,\{[(x + c)^2 + y^2]^{1/2} - [(x - c)^2 + y^2]^{1/2}\}, \tag{82f}$$

$$\frac{x^2}{c^2 \cosh^2 \xi} + \frac{y^2}{c^2 \sinh^2 \xi} = 1, \quad \frac{x^2}{c^2 \cos^2 \eta} - \frac{y^2}{c^2 \sin^2 \eta} = 1. \tag{82g, h}$$

If ξ is held constant while η is given all possible values, an ellipse is obtained on the z-plane, with semimajor and semiminor axes

$$a' = c \cosh \xi, \; b' = c\,|\sinh \xi|. \tag{82i, j}$$

The same ellipse is obtained for $\xi = -\xi_1$ as for $\xi = \xi_1$. If η is held constant while ξ ranges from $-\infty$ to ∞, a hyperbola is obtained with semiaxes

$$a'' = c\,|\cos \eta|, \; b'' = c\,|\sin \eta|. \tag{82k, l}$$

All ellipses and hyperbolas have common foci at $(\pm\, c, 0)$, and

$$a'^2 - b'^2 = c^2, \; a''^2 + b''^2 = c^2. \tag{82m, n}$$

See Figure 129, on which possible values of ξ and η are indicated in terms of $\pi/30$ as a unit.

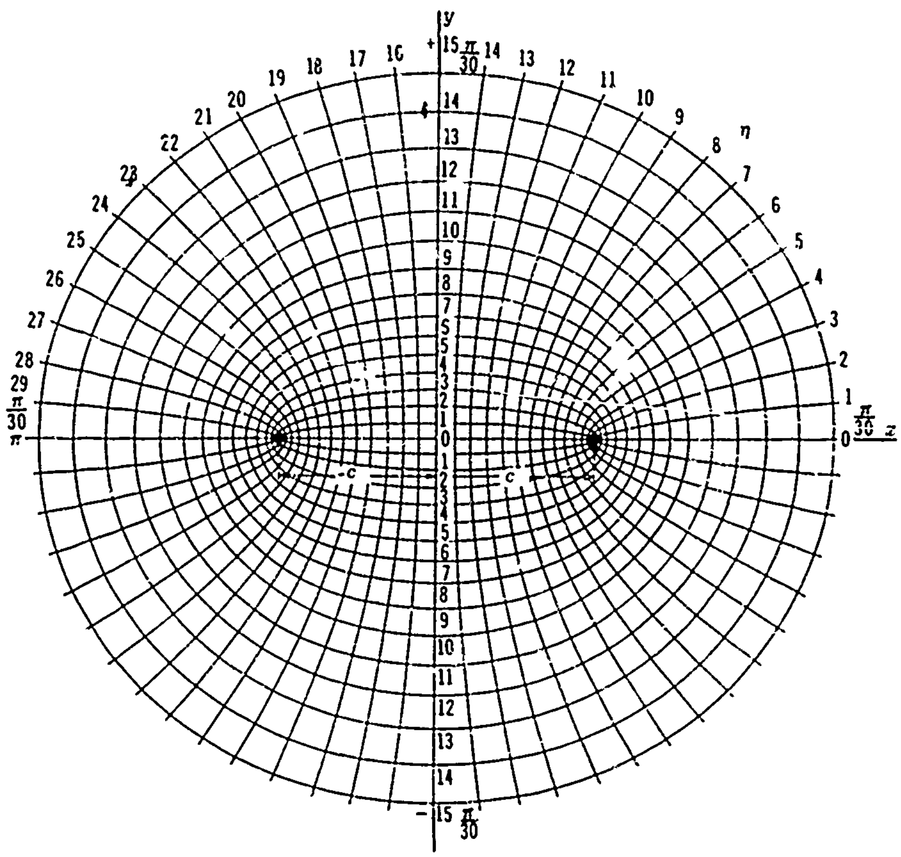


Figure 129 – Illustrating elliptic coordinates. See Section 82.

The ellipse for $\xi = 0$ reduces to the segment of the x-axis between $\pm c$, on which $x = c \cos \eta$. The remainder of the x-axis can be regarded as a hyperbola on which $\sin \eta = 0$ while $\cos \eta = 1$ for $x > c$ or $\cos \eta = -1$ for $x < -c$, and on which $x = \pm c \cosh \xi$.

The variables ξ, η can obviously be used as coordinates on the z-plane; they are called elliptic coordinates. They have the disadvantage of being doubly many-valued. Not only is η many-valued like an angle, with a period of 2π, but the values $-\xi$, $-\eta$ define the same point (x, y) as do ξ, η. If both ξ and η are required to vary continuously with x and y, ξ must change sign in crossing the x-axis between $x = \pm c$, since there $|\cos \eta| < 1$ and $\sin \eta \neq 0$, whereas in crossing at $|x| > c$, $|\xi| > 0$ and $\sin \eta$ must change sign with y. Hence it is easily seen that ξ does not change sign but η changes by $\pm 2\pi$ in going once around both of the points $(\pm c, 0)$; whereas, if only one of these points is encircled, upon returning to the starting-point, η has returned to its initial value but ξ has changed sign. In applications it is usually convenient to suppress at least the ambiquity as to ξ. The two most useful alternative conventions are the following.

(a) Keep $\xi \geq 0$. Then from Equation [82c, d] it is easily seen that, as the x-axis is crossed between the points $x = \pm c$, $\sin \eta$ must change sign discontinuously without change of $\cos \eta$; η itself may change discontinuously to $-\eta$, or to $2\pi n - \eta$ where n is an integer, positive or negative. Elsewhere η may vary continuously: in this case η will differ on the two halves of each hyperbola and is many valued; in going once around both of the points $(\pm c, 0)$ in the same direction, η, like a polar angle, changes by $\pm 2\pi$.

A possible choice to make η single valued is the range $-\pi < \eta \leqq \pi$. Then η changes sign discontinuously in crossing the x-axis wherever $|x| < c$. Values of ξ and η according to this convention are indicated in Figure 130a.

(b) As an alternative, ξ may be given everywhere the same sign as y. Then ξ will have opposite signs on the two halves of each ellipse and will change sign discontinuously at the x-axis where $|x| > c$, whereas η may be made to vary continuously and will then have a fixed value on each hyperbola. A possible range is $0 \leqq \eta \leqq \pi$. This latter convention is illustrated in Figure 130b, and in more detail in Figure 129.

In any case, if $d\eta = 0$, $dx = c \sinh \xi \cos \eta \, d\xi$ and $dy = c \cosh \xi \sin \eta \, d\xi$; if $d\xi = 0$, $dx = -c \cosh \xi \sin \eta \, d\eta$ and $dy = c \sinh \xi \cos \eta \, d\eta$.

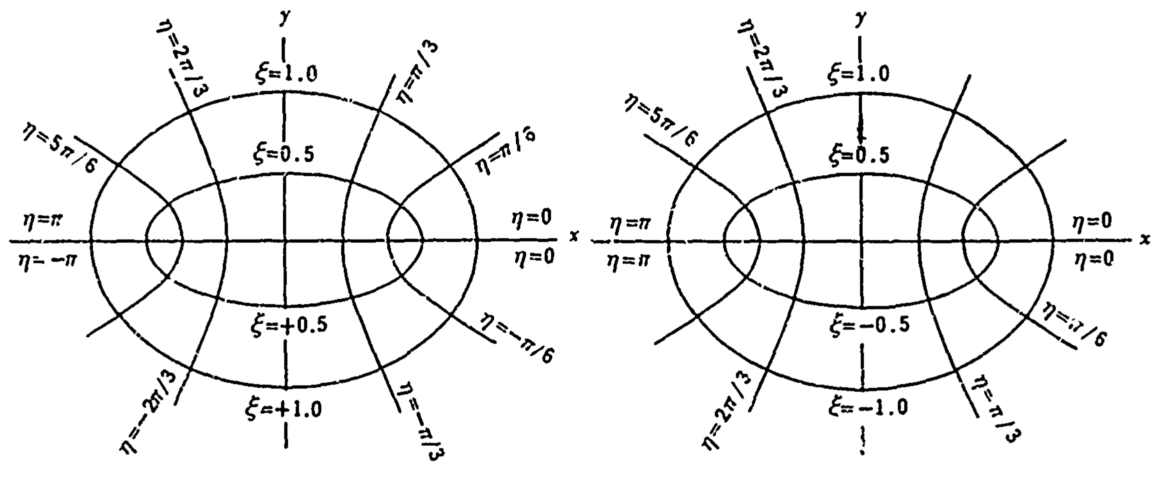


Figure 130a

Figure 130b

Figure 130 – Symbolism for flow parallel to major axis past an elliptic cylinder.

Hence the slope angles of the ξ and η coordinate directions are

$$\theta_\xi = \tan^{-1}\left(\frac{dy}{dx}\right)_{d\eta=0} = \tan^{-1}(\coth \xi \tan \eta); \quad \theta_\eta = \tan^{-1}\left(\frac{dy}{dx}\right)_{d\xi=0} = -\tan^{-1}(\tanh \xi \cot \eta). \quad [82o, p]$$

Also, the elements of distance in the coordinate directions, calculated as $ds = (dx^2 + dy^2)^{1/2}$ with either $d\eta = 0$ or $d\xi = 0$,

$$ds_\xi = cGd\xi,\ ds_\eta = cGd\eta, \qquad [82q, r]$$

$$G = (\sinh^2 \xi + \sin^2 \eta)^{1/2} = \left[\frac{1}{2} (\cosh 2\xi - \cos 2\eta)\right]^{1/2}. \qquad [82s]$$

by hyperbolic formulas in Section 32. Hence the components of velocity q_ξ and q_η in the coordinate directions are, from Equation [6f], in which ϕ denotes the velocity potential,

$$q_\xi = -\frac{1}{cG}\frac{\partial\phi}{\partial\xi},\ q_\eta = -\frac{1}{cG}\frac{\partial\phi}{\partial\eta}. \qquad [82t, u]$$

In applying these results it may be more convenient to substitute, in place of ξ, the semiaxes a, b, of the corresponding ellipse, always taken positive. Then, if always $\xi \geqq 0$, from Equations [82c, d] and [82i, j]

$$x = a'\cos\eta,\ y = b'\sin\eta,\ \xi = \ln[(a' + b')/c] \qquad [82v, w, x]$$

For any point (x, y), the value of a' can be found by adding distances from the foci and dividing by 2; then

$$b' = (a'^2 - c^2)^{1/2},\ \tan\eta = a'y/b'x.$$

The components of velocity, q_t along the tangent to the ellipse in the direction of increasing η, and q_n along the outward normal, are then, from Equations [82t, u] and [82j]

$$q_n = q_\xi = -\frac{1}{h'}\frac{\partial\phi}{\partial\xi},\quad q_t = q_\eta = -\frac{1}{h'}\frac{\partial\phi}{\partial\eta},\ h' = (b'^2 + c^2\sin^2\eta)^{1/2} \qquad [82y, z, a']$$

The components q_n and q_t make angles θ_t, θ_n with the x-axis where, from Equation [82o, p],

$$\theta_n = \theta_\xi = \tan^{-1}\left(\frac{a'}{b'}\tan\eta\right),\ \theta_t = \theta_\eta = -\tan^{-1}\left(\frac{b'}{a'}\cot\eta\right). \qquad [82b', c']$$

Geometrically, the transformation from z to ζ maps the entire z-plane continuously onto the positive half, and again on the negative half of a strip on the ζ-plane parallel to the ξ-axis and extending from $\eta = 0$ to $\eta = 2\pi$, as may be verified by consideration of the displacements on the ζ-plane that are required to reach all parts of the z-plane. The mapping is then repeated in this manner upon each successive parallel strip of width 2π.

If convention (a) is adopted for the values of ξ and η, with $\xi \geqq 0$ and $-\pi < \eta \leqq \pi$, the ring between two ellipses, or the interior of any ellipse, is mapped onto a rectangle with sides at $\eta = \pm\pi$ and with ends at the proper values of ξ. The ring may be supposed to be cut along the negative x-axis and straightened out. Using convention (b), the area between two hyperbolas is mapped onto an infinite strip parallel to the ξ-axis.

(See Reference 1, Article 71; Reference 1, Section 6.32.)

83. FLOW PAST AN ELLIPTIC CYLINDER

By means of the transformation discussed in Section 81, or $z = z'' + c^2/z''$, the flow around a stationary circular cylinder can be transformed into that around an elliptic cylinder. The appropriate complex potential w can be obtained by replacing $z - z_1$ by z'' in Equation [69j] and then substituting for z'' in terms of z. The result will be written down in a modified notation and verified. It is

$$w = U(a+b)\cosh(\zeta - \xi_0 - i\alpha) + \frac{i\Gamma}{2\pi}(\zeta - \xi_0), \qquad [83a]$$

$$z = x + iy = c\cosh\zeta,\ \zeta = \xi + i\eta,\ \xi \geqq 0, \qquad [83b, c]$$

where a and b are positive real constants, α and U are real constants, and

$$c = \sqrt{a^2 - b^2} > 0,\quad e^{\xi_0} = \frac{a+b}{c} = \frac{a+b}{a-b} \qquad [83d, e]$$

or

$$\sinh\xi_0 = \frac{b}{c},\ \cosh\xi_0 = \frac{a}{c} \qquad [83f, g]$$

Here ξ, η are the elliptic coordinates of Section 82.

Writing $\cosh\zeta = (e^{\zeta} + e^{-\zeta})/2$, it is found that

$$e^{\zeta} = \frac{2z}{c}\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{c^2}{z^2}}\right) = \frac{2z}{c} - \frac{c}{2z} \cdots,$$

$$e^{-\zeta} = \frac{2c}{z}\left(\frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{c^2}{z^2}}\right) = \frac{c}{2z} \cdots.$$

by use of the binomial expansion; hence, using also $\ln(1+x) = x - x^2/2 \ldots.$, writing $\cosh(\zeta - \xi_0 - i\alpha) = \frac{1}{2}(e^{\zeta - \xi_0 - i\alpha} + e^{-\zeta + \xi_0 + i\alpha})$, using Equation [83a, d, e], keeping only the first power of $1/z$,

$$\zeta = \ln\left[\frac{2z}{c}\left(1 - \frac{c^2}{4z^2}\ \ldots.\right)\right] = \ln\frac{2z}{c} - \frac{c^2}{4z^2}\ \ldots.,$$

$$w = Uze^{-i\alpha} + \frac{i\Gamma}{2\pi}\left(\ln z + \ln\frac{2}{c} - \xi_0\right)$$

$$+ U(b\cos\alpha + ia\sin\alpha)\frac{a+b}{2z} + \ldots. \qquad [83h]$$

Thus at infinity $w \to Uze^{-i\alpha}$ and represents a stream approaching at velocity U from a direction making an angle α with the positive x-axis; see Section 35 and Figure 131.

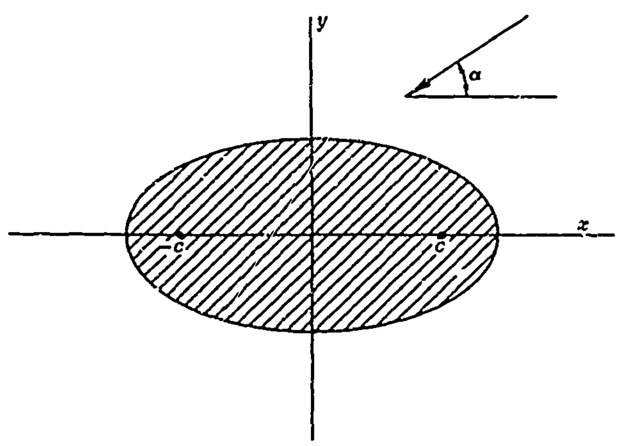


Figure 131 – Symbolism for flow past an elliptic cylinder.

The variables ξ, η are the elliptic coordinates described in the last section. They are related to x and y by Equations [82c, d] or [82v, w]. The ξ curves are confocal ellipses with foci at $(\mp c, 0)$. Here $\xi > 0$.

Using the hyperbolic formulas listed in Section 32, from $w = \phi + i\psi$,

$$\phi = U(a+b)\cosh(\xi - \xi_0)\cos(\eta - \alpha) - \frac{\Gamma\eta}{2\pi}, \qquad [83i]$$

$$\psi = U(a+b)\sinh(\xi - \xi_0)\sin(\eta - \alpha) + \frac{\Gamma}{2\pi}(\xi - \xi_0), \qquad [83j]$$

or, from Equations [83f, g], [82i, j], [83d, e] and [82x],

$$\phi = \frac{U}{a-b}(a'a-b'b)\cos(\eta-\alpha) - \frac{\Gamma\eta}{2\pi}, \qquad [83k]$$

$$\psi = \frac{U}{a-b}(b'a-a'b)\sin(\eta-\alpha) + \frac{\Gamma}{2\pi}\ln\frac{a'+b'}{a+b}. \qquad [83l]$$

Thus $\psi = 0$ on the ellipse $\xi = \xi_0$, whose equation from Equations [82g] and [83f, g], is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

This ellipse may represent the profile of a solid elliptic cylinder immersed in the fluid stream, with its major axis parallel to the flow at infinity. In going once around the cylinder, η increases by 2π and ϕ decreases by Γ. Hence there is circulation Γ around the cylinder.

If $\Gamma = 0$, the remainder of the streamline for $\psi = 0$ is defined by $\eta = \alpha$ on the forward side or $\eta = \alpha + \pi$ on the rear side; it consists of hyperbolic arcs.

The components of velocity at any point (x, y), respectively tangential and normal to the ξ-ellipse that passes through (x, y), or in the directions specified in Equations [82b′, c′], are, from Equations [82y, z] and [82i, j], [83f, g],

$$q_n = q_\xi = -\frac{U(b'a-a'b)}{(a-b)(b'^2+c^2\sin^2\eta)^{1/2}}\cos(\eta-\alpha), \qquad [83m]$$

$$q_t = q_\eta = \frac{1}{(b'^2+c^2\sin^2\eta)^{1/2}}\left[U\frac{a'a-b'b}{a-b}\sin(\eta-\alpha) + \frac{\Gamma}{2\pi}\right] \qquad [83n]$$

On the x-axis, where $\eta = 0$ or π and $x = \pm a'$, respectively, $b' = \sqrt{x^2-c^2}$,

$$u = \pm q_n = -\frac{U}{a-b}\left(a - \frac{b|x|}{\sqrt{x^2-c^2}}\right)\cos\alpha, \qquad [83o]$$

$$v = \pm q_t = -\frac{U}{a-b}\left(\frac{a|x|}{\sqrt{x^2-c^2}} - b\right)\sin\alpha \pm \frac{\Gamma}{2\pi\sqrt{x^2-c^2}} \qquad [83p]$$

On the y-axis, where $\eta = \pi/2$ or $3\pi/2$, $y = \pm b'$, $a' = \sqrt{b'^2+c^2} = \sqrt{y^2+c^2}$,

$$u = \mp q_t = -\frac{U}{a-b}\left(a - \frac{b\,|y|}{\sqrt{y^2+c^2}}\right)\cos\alpha \mp \frac{\Gamma}{2\pi\sqrt{y^2+c^2}} \qquad [83q]$$

$$v = \pm q_n = -\frac{U}{a-b}\left(\frac{a\,|y|}{\sqrt{y^2+c^2}} - b\right)\sin\alpha\,. \qquad [83r]$$

On the cylinder itself $a' = a$, $b' = b$, $x = a\cos\eta$, $y = b\sin\eta$, hence $q_n = 0$,

$$q_t = \frac{b}{(b^4 + c^2 y^2)^{1/2}}\left[U\,(a+b)\left(\frac{y}{b}\cos\alpha - \frac{x}{a}\sin\alpha\right) + \frac{\Gamma}{2\pi}\right] \qquad [83s]$$

and $q = |q_t|$. If $\Gamma = 0$, stagnation points occur where $y/x = (b/a)\tan\alpha$. For comparisons with experiment, see Zahm, References 101 and 102.

Examples of the streamlines for $\Gamma = 0$ and $\alpha = 0$ deg, 45 deg, and 90 deg are shown in Figures 132, 133, and 134. Here $a/b = 2$, $\xi_0 = \coth^{-1} 2 = 0.549$. In two cases only half of the symmetrical diagram is shown. In two cases the excess of pressure above that at infinity is shown, for steady motion, at points on the axes or on the cylinder, by curves labeled $p - p_\infty$. For points on the y-axis, $p - p_\infty$ is plotted horizontally from the y-axis as a base with positive values toward the right. In Figure 135 the calculated pressure on an elliptic cylinder with $\Gamma = \alpha = 0$, represented by the broken curve, is compared with observed values in air at 40 miles per hour, which are represented by small circles (from Reference 101).

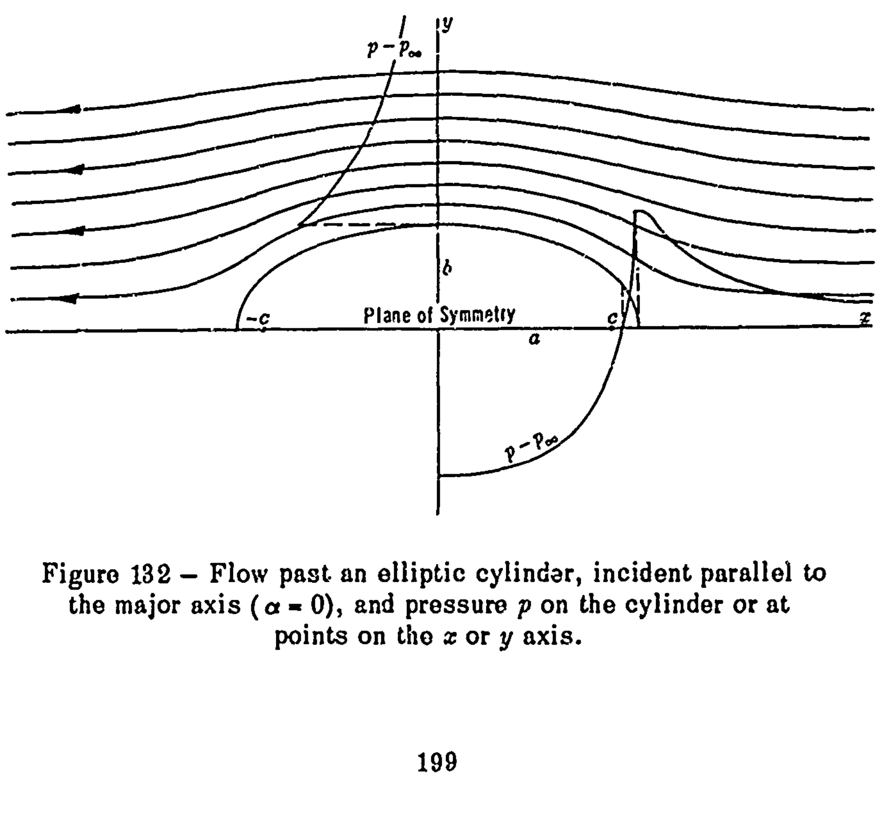


Figure 132 – Flow past an elliptic cylinder, incident parallel to the major axis ($\alpha = 0$), and pressure p on the cylinder or at points on the x or y axis.

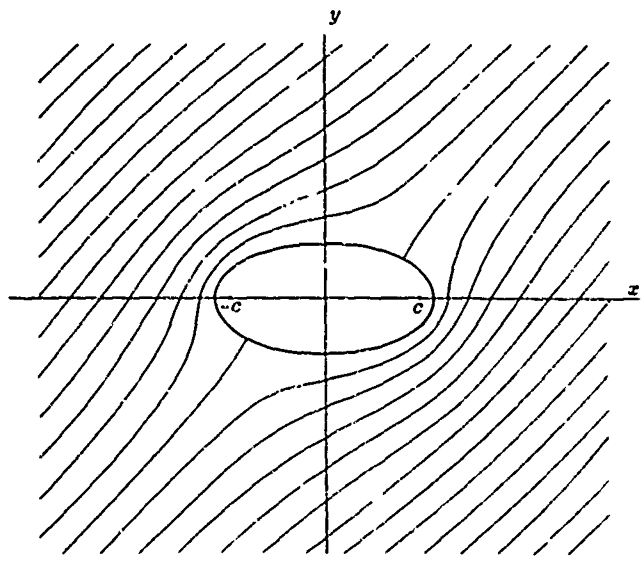


Figure 133 – Flow past an elliptic cylinder with $\alpha = 45$ degrees.

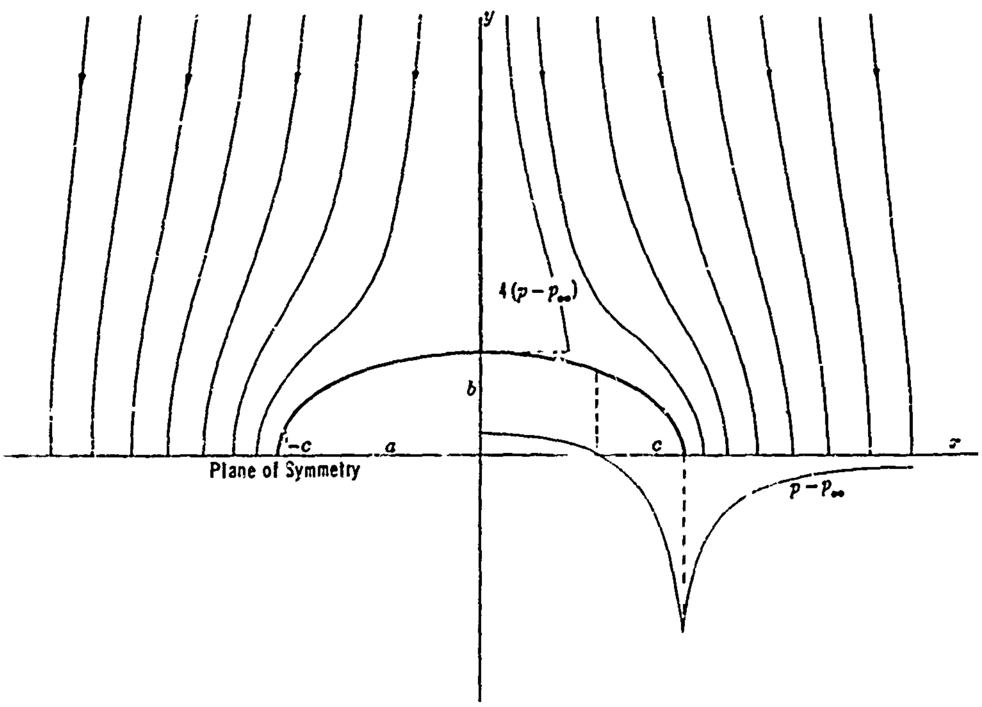


Figure 134 – Flow past an elliptic cylinder, incident parallel to the minor axis ($\alpha = 90°$), and pressure p at points on the cylinder or on the x or y axis.

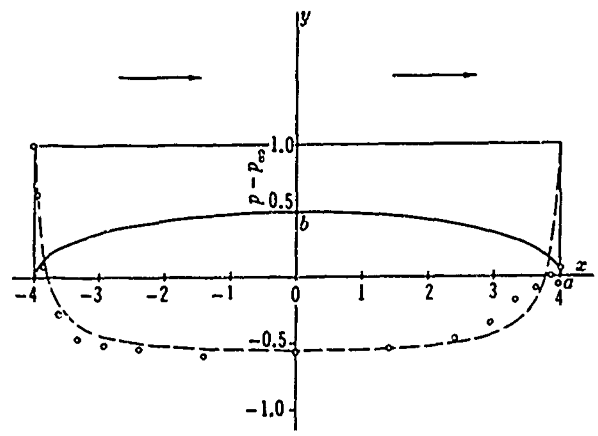


Figure 135 – On an elliptic cylinder the calculated pressure is shown by a broken curve and observed pressures by small circles.

In steady motion the resultant force is a lift $\rho\Gamma U$ per unit length, according to the Kutta-Joukowski theorem proved in Section 73. Furthermore, comparison of Equation [83h] with Equation [74h] shows that here $\gamma = \alpha$, $b_1 = U(a+b)(b \cos \alpha + ia \sin \alpha)/2$; hence Equation [74k] gives for the torque per unit length on the cylinder about an axis through the origin, in steady motion,

$$N = -\frac{1}{2} \pi\rho (a^2 - b^2) U^2 \sin 2\alpha. \qquad [83t]$$

Because of the sign, the torque tends to set the cylinder broadside to the stream.

An elliptic cylinder in a *converging* stream was considered by Oka[107].

(For notation and method; see Section 34; Reference 1, Section 71; Reference 2, Sections 6.31, 6.32, 6.33, 6.42; Zahm and others, References 182 and 101.)

84. ELLIPTIC CYLINDER IN TRANSLATION

Let the cylinder described in the last section be itself in motion at velocity U in a direction inclined at an angle α to the positive x-axis or to the major axis of the ellipse, and let the surrounding fluid be at rest at infinity. This case can be produced out of the preceding by imposing on everything a uniform velocity U in the required direction. Then, from Equations [35a] and [82a], there is to be added in w the term

$$-Uze^{-i\alpha} = -cUe^{-i\alpha} \cosh \zeta.$$

After inserting exponentials in place of all hyperbolic cosines and eliminating ξ_0 and c by means of Equation [83d, e], from Equation [83a],

$$w = U\sqrt{\frac{a+b}{a-b}}\,(b \cos \alpha + ia \sin \alpha)e^{-\zeta} + \frac{i\Gamma}{2\pi}(\zeta - \xi_0), \qquad [84a]$$

$$\phi = U\sqrt{\frac{a+b}{a-b}}\,(b\cos\alpha\ \cos\eta + a\sin\alpha\ \sin\eta)e^{-\xi} - \frac{\Gamma\eta}{2\pi}, \qquad [84b]$$

$$\psi = -U\sqrt{\frac{a+b}{a-b}}\,(b\cos\alpha\ \sin\eta - a\sin\alpha\ \cos\eta)e^{-\xi} + \frac{\Gamma}{2\pi}(\xi - \xi_0), \qquad [84c]$$

or, by Equations [82x] and [83d, e],

$$\phi = U\,\frac{a+b}{a'+b'}\,(b\cos\alpha\ \cos\eta + a\sin\alpha\ \sin\eta) - \frac{\Gamma\eta}{2\pi}, \qquad [84d]$$

$$\psi = -U\,\frac{a+b}{a'+b'}\,(b\cos\alpha\ \sin\eta - a\sin\alpha\ \cos\eta) + \frac{\Gamma}{2\pi}\ln\frac{a'+b'}{a+b} \qquad [84e]$$

These formulas hold at any instant provided the axes are drawn with the origin on the axis of the cylinder and the x-axis along the major axis of its profile. The surface of the cylinder is the ellipse $\xi = \xi_0$, or $a' = a$, $b' = b$.

The components of velocity, in directions given by Equation [82b', c'], are, from Equations [82y, z], [84b] and [82x],

$$q_n = q_\xi = U\,\frac{a+b}{a'+b'}\,\frac{b\cos\alpha\ \cos\eta + a\sin\alpha\ \sin\eta}{(b'^2 + c^2\sin^2\eta)^{1/2}}, \qquad [84f]$$

$$q_t = q_\eta = \frac{1}{(b'^2 + c^2\sin^2\eta)^{1/2}}\left[U\,\frac{a+b}{a'+b'}\,(b\cos\alpha\ \sin\eta - a\sin\alpha\ \cos\eta) + \frac{\Gamma}{2\pi}\right] \qquad [84g]$$

On the cylinder itself $a' = a$, $b' = b$.

On the x-axis,

$$\eta = 0 \text{ or } \pi,\ x = \pm a',\ b' = \sqrt{a'^2 - c^2} = \sqrt{x^2 - c^2},$$

and

$$u = \pm q_n = \frac{U b\,(a+b)\cos\alpha}{\sqrt{x^2 - c^2}\,(|x| + \sqrt{x^2 - c^2})}, \qquad [84h]$$

$$v = \pm q_t = -\frac{U a\,(a+b)\sin\alpha}{\sqrt{x^2 - c^2}\,(|x| + \sqrt{x^2 - c^2})} + \frac{\Gamma}{2\pi\sqrt{x^2 - c^2}}. \qquad [84i]$$

On the y-axis,

$$\eta = \pi/2 \text{ or } 3\pi/2,\ y = \pm b',\ a' = \sqrt{y^2 + c^2}$$

and

$$u = \mp q_t = -\frac{U b (a + b) \cos \alpha}{\sqrt{y^2 + c^2} (|y| + \sqrt{y^2 + c^2})} , \quad [84j]$$

$$v = \pm q_n = \frac{U a (a + b) \sin \alpha}{\sqrt{y^2 + c^2} (|y| + \sqrt{y^2 + c^2})} . \quad [84k]$$

The formulas for this case and the preceding are readily shown to differ by terms representing a uniform flow, with use of the fact that $(a + b)(a - b) = c^2 = (a' + b')(a' - b')$.

If there is no circulation about the cylinder, $\Gamma = 0$.

For motion parallel to the major axis, $\alpha = 0$ or π; to the minor axis, $\alpha = \pi/2$ or $3\pi/2$. If $\Gamma = 0$, the geometrical flow net is the same for flow parallel to either axis; the ϕ curves for one case become the ψ curves for the other, and ϕ, ψ and all velocities are changed in a uniform ratio. The general case, for which the formulas have been written, can be regarded as formed by the superposition of these two simpler cases.

Furthermore, if the motion is parallel to an axis, and if $\Gamma = 0$, the velocity at a given external point is the same for all confocal forms of the cylinder. For the relations between x, y, and ξ, η are unaffected so long as the foci are not disturbed; and changing a and b merely multiplies ϕ, ψ and hence all velocities by a uniform factor.

Figure 138 will serve to illustrate the flow for motion parallel to either axis. The ellipse drawn as a broken curve, or any other ellipse confocal with it, may represent the cylinder. The foci are at the ends of the horizontal heavy line. Either family of curves, that crossing the vertical or the horizontal axis, constitutes streamlines according as the motion is parallel to the major or to the minor axis; the curves of the other set are then the equipotentials. The arrows on the curves refer to motion along the minor axis.

No similar identities occur in motion oblique to the axes.

The kinetic energy of the fluid, per unit length of the cylinder is, by Equation [17d], when $\Gamma = 0$,

$$T_1 = -\frac{\rho}{2} \int \phi \, d\psi = \frac{\rho U^2}{2} \; \frac{a + b}{a - b} \; e^{-2\xi_0} \int_0^{2\pi} (b \cos \alpha \; \cos \eta + a \sin \alpha \; \sin \eta)^2 \, d\eta$$

or, using Equation [83d, e],

$$T_1 = \frac{\pi}{2} \rho U^2 (b^2 \cos^2 \alpha + a^2 \sin^2 \alpha). \quad [84l]$$

The forces are as in the last case, Section 83.

(For notation and method; see Section 34; Reference 1, Article 71; Reference 2, Section 9.65; Ratib[104]; Kr̈enes[105].)

85. FLOW PAST A PLANE LAMINA

If $b \to 0$, the cylinder of the last two sections becomes a plane lamina of width $2a$, with its edges at $(\pm a,0)$, on which a stream impinges at velocity U and at an angle of inclination α to its faces. Then $c = a$, $\xi_0 = 0$. The general formulas need not be repeated, but a few points may be noted.

On the lamina itself, $a' = a = c$, $b' = b = 0$, and from $x = a \cos \eta$ and Equation [83n], after expanding $\sin (\eta - \alpha)$,

$$u = \mp q_t = U\left(-\cos\alpha \pm \frac{x}{\sqrt{a^2 - x^2}} \sin\alpha\right) \mp \frac{\Gamma}{2\pi\sqrt{a^2 - x^2}}, \qquad [85a]$$

where the upper sign refers to the front face, on which $0 < \eta < \pi$, and the lower sign to the back face; and $q = |u|$.

Thus $q \to \infty$ at the edges of the lamina, in general. By assigning the proper value to Γ, however, q may be made finite at one edge. Thus, if $\Gamma = -2\pi aU \sin\alpha$, u approaches

$$-U\cos\alpha \text{ as } x \to -a; \text{ for, } (x + a)/\sqrt{a^2 - x^2} = [(a + x)/(a - x)]^{1/2} \to 0 \text{ as } x \to -a.$$

If $\Gamma = 0$, stagnation points occur on the lamina at $\eta = \alpha$ and at $\eta = \alpha + \pi$, or at $x = a \cos\alpha$ on the front face and at $x = -a\cos\alpha$ on the rear face. The hyperbolic dividing streamline meets the lamina at the first of these points and leaves it at the second; the two hyperbolic arcs, with foci at the edges, are asymptotic to a line drawn through the center of the lamina and inclined at an angle α to its plane. On the front face, $u = U$ at $x = a\cos(\alpha/2)$ and $u = -U$ at $x = -a\sin(\alpha/2)$.

When $\Gamma = 0$, however, a more direct formulation becomes possible. Then, from Equation [83a] with $b = 0$, $w = aU\cosh(\zeta - i\alpha) = aU\cosh\zeta\cos\alpha - iaU\sinh\zeta\sin\alpha$. Here, since $c = a$, the term in $\cos\alpha$ equals $Uz\cos\alpha$ and so represents a uniform flow parallel to the plane of the lamina, which need not be further considered.

The term in $\sin\alpha$ taken by itself represents a stream flowing toward negative y and impinging perpendicularly on the disk. Dropping for a moment the proportionality factor $\sin\alpha$, so that the velocity of the stream at infinity is U, its complex potential is

$$w = -iaU\sinh\zeta = -iU(z^2 - a^2)^{1/2}. \qquad [85b]$$

If $w = \phi + i\psi$,

$$\phi^2 - \psi^2 = U^2(a^2 + y^2 - x^2), \quad \phi\psi = -U^2 xy. \qquad [85c,d]$$

These equations can easily be solved, either for ϕ and ψ, or for x and y. The signs of ϕ and ψ may be inferred either from physical considerations or from a detailed study of Equation [85b].

On the y-axis; $\psi = 0$, $\phi = U\sqrt{a^2 + y^2}$; $u = 0$, $v = -Uy\sqrt{a^2 + y^2}$.

On the x-axis where $|x| < a$: $\psi = 0$, $\phi = \pm U\sqrt{a^2 - x^2}$, also $v = 0$, $u = \pm Ux/\sqrt{a^2 - x^2}$, where the upper sign refers to the front face and the lower sign to the rear face. Thus $q = |u| = |U|$ at $|x| = a/\sqrt{2}$.

On the x-axis where $|x| > a$: $\phi = 0$, $\psi = \mp U\sqrt{x^2 - a^2}$, where the sign $\mp$ is opposite to the sign of x.

In this case, where the y-axis is a streamline and may represent an infinite rigid surface, half of the flow may represent a stream flowing past a straight boundary carrying a straight rigid stiffener of width a and negligible thickness, perpendicular both to the boundary and to the stream.

Two cases for $\Gamma = 0$, with $\alpha = 45$ deg and $\alpha = 90$ deg, respectively, are shown in Figures 136 and 137. In the first figure, the x-axis is rotated into a convenient direction. The points at which $u = \pm U$ are shown by short marks.

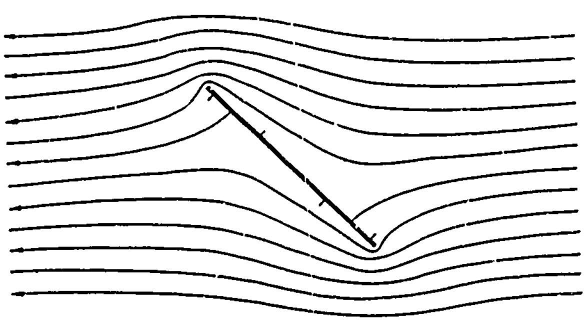


Figure 136 – Flow past a plane lamina in a direction inclined at 45° to the lamina.

Rotation of the stream and lamina through –90 deg, so that U (if positive) is directed toward negative x and the lamina lies along the y-axis, gives, perhaps by using Equation [25k] with $\alpha = -90$ deg, $k = 1$ and $h = 0$, $w = U(z^2 + a^2)^{1/2}$ and

$$\phi^2 - \psi^2 = U^2(x^2 + a^2 - y^2),\ \phi\psi = U^2 xy.$$

In steady motion, the lift on the lamina is in any case $\rho\Gamma U$, as on the cylinder, and the torque on it, from Equation [83t], is

$$N = -\frac{1}{2}\pi\rho a^2 U^2 \sin 2\alpha \qquad [85e]$$

where α is the angle between its direction of motion and the plane of its faces. The torque tends to set the lamina at right angles to the stream.

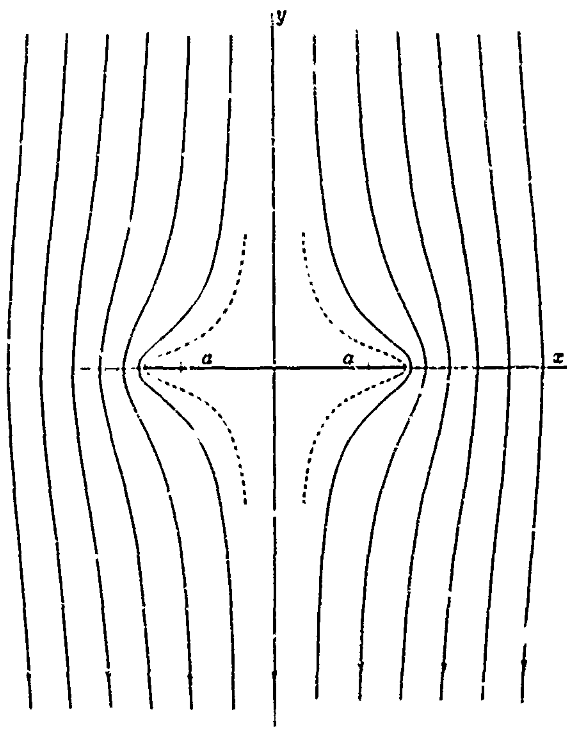


Figure 137 – Flow past a plane lamina at $\alpha = 90$ degrees.

It may seem strange that the lift should remain perpendicular to U at oblique angles, although the pressure on the lamina is everywhere perpendicular to its faces. The explanation lies in the occurrence of infinite velocities at the edges. In such cases erroneous results may be obtained if the forces are calculated from an integration of the pressures. In the present case, study of the behavior of the pressure distribution over the ellipsoid as it becomes progressively flattened into a lamina indicates that finite forces must be supposed to act on the edges of the lamina; see Morton, Reference 106. Mathematically, the limit of the integral giving the lift on the ellipsoid is not the same as the integral of the limit of the integrand, which represents pressure on the lamina. That the limit of the force must be the correct value for the lamina, on the other hand, is physically obvious, since no discontinuous change occurs in the motion of the neighboring fluid as the ellipsoid is flattened.

(For notation and method; see Section 34; Reference 1, Article 71.)

86. PLANE LAMINA IN TRANSLATION.

If the plane lamina described in the last section moves in translation through fluid at rest at infinity, only its perpendicular component of motion is significant, since motion parallel to its plane does not disturb the fluid. Let the lamina lie parallel to the x-axis and be moving toward positive y at velocity U. The relevant formulas may then be obtained from those of Section 84 by substituting $b = 0$, $\xi_0 = 0$, $c = a$, $\alpha = \pi/2$.

On the lamina itself $b' = 0$, $a' = a$, $x = a \cos \eta$ by Equation [82c], hence, from Equation [84f, g], $q_n = U$ and

$$u = \mp q_t = \pm \frac{1}{\sqrt{a^2 - x^2}} \left(Ux - \frac{\Gamma}{2\pi} \right). \qquad [86a]$$

where the upper sign refers to the front face and the lower to the rear face. If $\Gamma = \pm 2\pi a U$, the velocity is finite at one edge.

The flow net for $\Gamma = 0$ is shown in Figure 138, in which the dotted ellipse is now to be ignored. The arrows have reference to motion upward.

The kinetic energy per unit length is, from Equation [84i],

$$T_1 = \frac{\pi}{2} \rho a^2 U^2. \qquad [86b]$$

The forces are as in the last section; see Reference 1, Article 71.

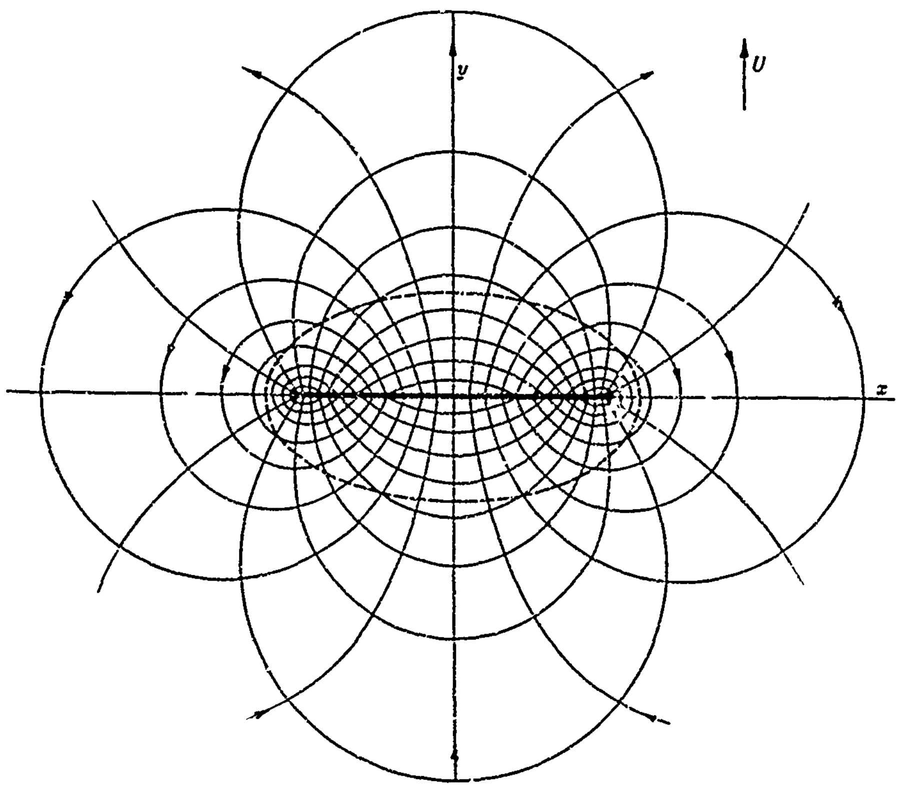


Figure 138 – Flow net around a plane lamina moving perpendicularly at velocity U with $\Gamma = 0$.

87. PARABOLIC CYLINDERS

Consider the transformation

$$\lambda = i\sqrt{2z} = \lambda_1 + i\lambda_2, \quad z = x + iy, \qquad [87a, b]$$

$$x = \frac{1}{2}(\lambda_2^2 - \lambda_1^2), \quad y = -\lambda_1\lambda_2; \quad \lambda_1^2 = r - x, \quad \lambda_2^2 = r + x, \qquad [87c, d, e, f]$$

$$r = \sqrt{x^2 + y^2}. \qquad [87g]$$

The surfaces λ_1 = constant, or λ_2 = constant, constitute two families of orthogonal confocal parabolas, with the x-axis as their axis and the focus at the origin; they open toward $x \to \infty$ and $x \to -\infty$, respectively; see Figure 139. The parabola for $\lambda_1 = 0$ is the positive x-axis, that for $\lambda_2 = 0$ the negative.

The variables λ_1, λ_2 may be used as *parabolic coordinates* on the xy-plane. They are double valued, and changes of sign of λ_1 or λ_2 on the same parabola are necessary in order to cover the entire plane.

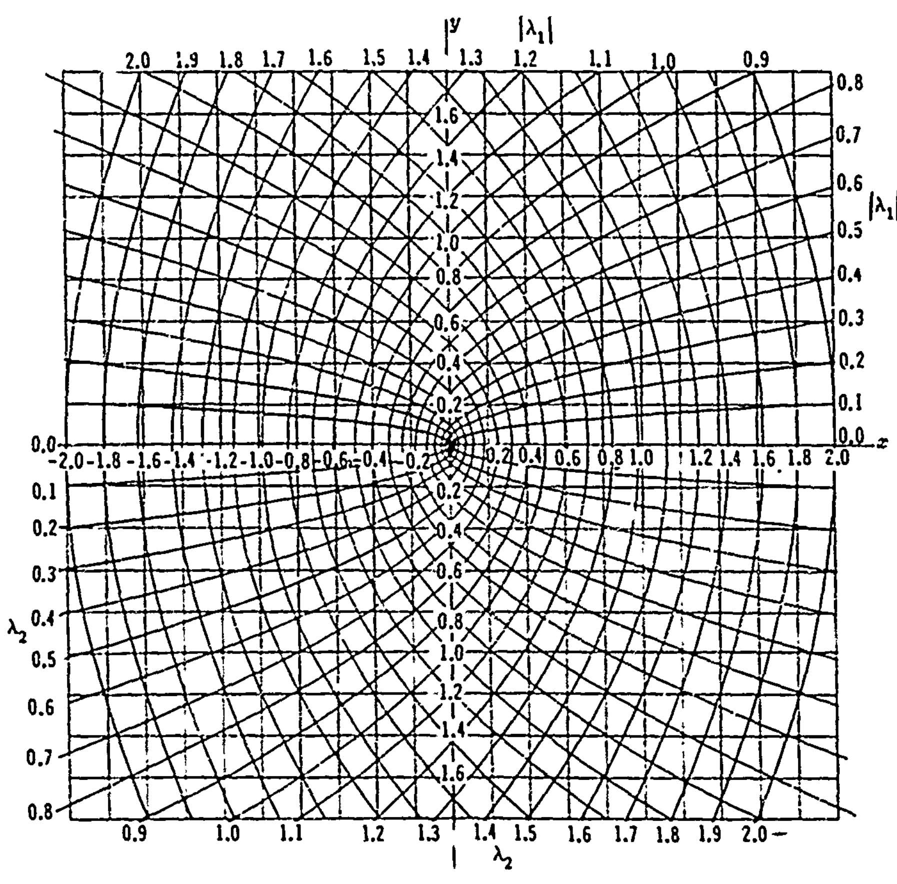


Figure 139 – Diagram for parabolic coordinates.

Circulatory Flow. If $w = A\lambda$, $\phi = A\lambda_1$, $\psi = A\lambda_2$, where A is real, the λ_2 parabolas become streamlines in a type of flow in which there is a singularity at the origin and the velocity vanishes at infinity. For, then

$$q = \left|\frac{dw}{dz}\right| = \left|\frac{dw}{d\lambda}\Big/\frac{d\lambda}{dz}\right| = \left|\frac{iA}{\sqrt{2z}}\right| = \frac{|A|}{\sqrt{2r}} \,. \qquad \text{[illegible]}$$

This might represent a sort of circulatory flow past a parabolic cylinder whose cross-sectional profile is represented by one of the λ_2 parabolas, or between two such cylinders corresponding to two values of λ_2.

Streaming Flow. Let

$$w = -U\left(\frac{1}{2}\lambda^2 - i\beta\lambda\right), \; U \text{ and } \beta \text{ real and } \beta > 0, \qquad \text{[87i]}$$

$$\phi = -U\left[\frac{1}{2}(\lambda_1^2 - \lambda_2^2) + \beta\lambda_2\right] = U(x - \beta\sqrt{r+x}) \qquad \text{[87j]}$$

$$\psi = -U\lambda_1(\lambda_2 - \beta) = U(y \mp \beta\sqrt{r-x}) = Uy\left(1 - \frac{\beta}{\sqrt{r+x}}\right). \qquad \text{[87k]}$$

Here, in accord with the labeling in Figure 139, continuity has been secured, except on the negative x-axis, by assuming that $\lambda_2 \geqq 0$; then the sign of λ_1 and the sign before $\sqrt{r-x}$ must be taken opposite to the sign of y, but $\sqrt{r+x}$ is positive. For the velocity

$$u = U\left(-1 + \frac{\beta}{2r}\sqrt{r+x}\right), \quad v = \frac{\beta U y}{2r\sqrt{r+x}}, \qquad \text{[87l, m]}$$

$$q^2 = U^2\left(1 - \frac{\beta}{r}\sqrt{r+x} + \frac{\beta^2}{2r}\right). \qquad \text{[87n]}$$

Thus toward infinity $v \to 0$, $u \to -U$, and the flow becomes a uniform stream at velocity U toward negative x. On the positive x-axis $r = x$ and

$$u = -U\left(1 - \frac{\beta}{\sqrt{2x}}\right). \qquad \text{[87o]}$$

The value $\psi = 0$ occurs on the positive x-axis, where $\lambda_1 = 0$, on the parabola at $\lambda_2 = \beta$. On this parabola $\sqrt{r+x} = \beta$ so that its apex, which represents the stagnation line, is at $x = \beta^2/2$; its semi-latus-rectum is the value of $|y|$ when $x = 0$ and $r = y$, or β^2. A solid cylinder may be inserted along this parabola. On the cylinder

$$q^2 = U^2 \left(1 - \frac{\beta^2}{2r}\right). \qquad [87p]$$

Streamlines for the flow past such a parabolic cylinder are shown in Figure 140; only half of the symmetrical plot is shown. The excess of pressure above that at infinity, in steady motion, is also shown as $p - p_\infty$, for points on the cylinder or on the x-axis ahead of it. This excess is everywhere positive; on the cylinder, it is $\rho U^2 \beta^2/4r$, on the x-axis ahead of it,

$$p - p_\infty = \rho U^2 \left(\frac{\beta}{\sqrt{2x}} - \frac{\beta^2}{4x}\right). \qquad [87q]$$

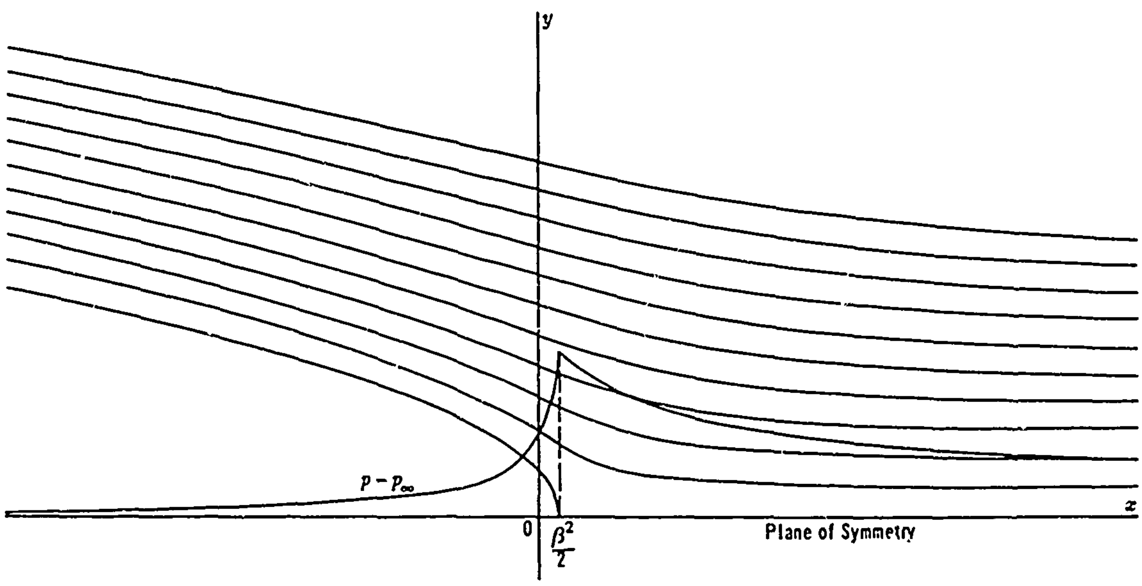


Figure 140 – Flow past a parabolic cylinder.

All such flow nets are similar, differing only in scale or in position; for, if β is changed, it is only necessary to change x, y, w, ϕ and ψ in proportion to β^2 in order to have all equations satisfied.

(For notation and method; see Section 34.)

88. THE CIRCULAR-ARC TRANSFORMATION

Equation [78a] of Section 78 invites generalization as follows:

$$\frac{z' - nc}{z' + nc} = \left(\frac{z - c}{z + c}\right)^n \qquad [88a]$$

where n and c are positive real numbers. The points $z = \pm c$ now correspond to $z' = \pm nc$, and at these points, in general, comformality fails.

Writing, as illustrated in Figure 141,

$$z - c = r_1 e^{i\theta_1},\ z + c = r_2 e^{i\theta_2},\ z' - nc = r_1' e^{i\theta_1'},\ z' + nc = r_2' e^{i\theta_2'},$$

where $-\pi \leq \theta_1 \leq \pi,\ -\pi \leq \theta_2 \leq \pi,$
it follows that

$$\theta_1' - \theta_2' = n(\theta_1 - \theta_2). \qquad [88b]$$

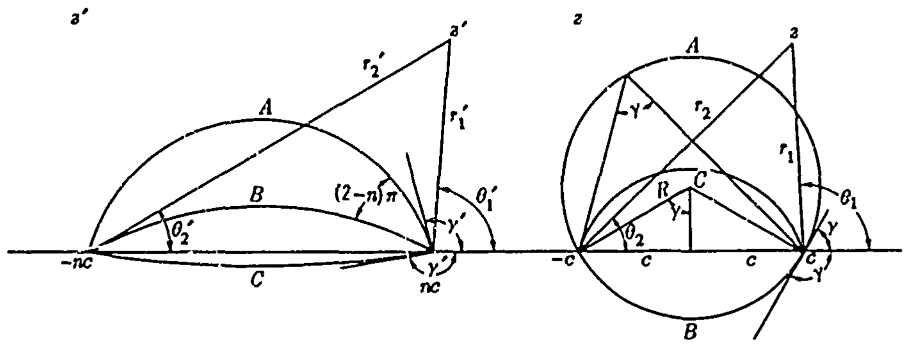


Figure 141 – Illustration for a circular arc A, B or C. See Section 88.

This shows that any circular arc joining $z = \pm c$, along which $\theta_1 - \theta_2$ has a constant value, transforms into one joining $z' = \pm nc$. The tangent to the z arc makes an external angle $\gamma = \theta_1 - \theta_2$ with its chord produced beyond $z = c$, or with the positive x-axis; the tangent to the z' arc makes a similar angle γ' with its chord where

$$\gamma' = n\gamma. \qquad [88c]$$

Here $-\pi \leq \gamma \leq \pi$. The respective radii of the arcs, which subtend angles $2\pi - 2\gamma$ or $2\pi - 2\gamma'$ at their centers, are $R = c/|\sin\gamma|$, $R' = c/|\sin\gamma'|$.

To solve for x' and y', where $x' + iy' = z'$, write

$$\zeta = \ln\frac{z + c}{z - c} = \lambda - i\mu,\ \lambda = \ln\frac{r_2}{r_1},\ \mu = \theta_1 - \theta_2. \qquad [88d,e,f]$$

Then $(z' + nc)/(z' - nc) = e^{n\zeta}$ and, solving for z',

$$z' = nc\coth\frac{n\zeta}{2}, \qquad [88g]$$

$$x' = \frac{nc}{2g'} \sinh n\lambda, \; y' = \frac{nc}{2g'} \sin n\mu, \tag{88h, i}$$

$$g' = \sinh^2 \frac{n\lambda}{2} + \sin^2 \frac{n\mu}{2} = \frac{1}{2} (\cosh n\lambda - \cos n\mu), \tag{88j}$$

from hyperbolic formulas listed in Section 32. Similarly

$$z = c \coth \frac{\zeta}{2}, \tag{88k}$$

$$x = \frac{c}{2g} \sinh \lambda, \;\; y = \frac{c}{2g} \sin \mu, \; g = \frac{1}{2} (\cosh \lambda - \cos \mu). \tag{88l, m, n}$$

Thus μ has the sign of y and $-\pi \leqq \mu \leqq \pi$. The variables λ, μ are sometimes called bipolar coordinates on the z-plane.

The transformation can be visualized by imagining the z-plane to be cut along the real axis between $\pm\, c$ and to be pulled or pushed until all arcs come into the proper position, with the remainder of the real axis retaining its direction.

An important special case is that of a circle through $(\pm\, c, 0)$ on the z-plane, such as AB, which transforms into two arcs meeting at an interior angle $2\pi - \Sigma|\gamma|$ or $(2 - n)\, \pi$. If $n = 2$, these coalesce into a single arc, as in Section 78. If $0 < n < 2$, the exterior of the circle is mapped conformally onto the part of the z'-plane lying outside of the crescent enclosed by the two arcs. If $0 < n < 1$, the ends of the "crescent" are reentrant; compare Figure 142b.

Toward infinity, Equation [88a] becomes, by binomial expansion,

$$\left(1 - \frac{nc}{z'} \ldots\right) \Big/ \left(1 + \frac{nc}{z'} \ldots\right) = \left(1 - \frac{nc}{z} \ldots\right) \Big/ \left(1 + \frac{nc}{z} \ldots\right).$$

Hence at infinity $z' \to z$ and the two planes agree.

The transformation fails to be conformal, in general, at $z = \pm\, c$. From Equations [88g] and [88k]

$$\frac{dz'}{dz} = \frac{dz'}{d\zeta} \left(\frac{dz}{d\zeta}\right)^{-1} = n^2 \sinh^2 \frac{\zeta}{2} \Big/ \sinh^2 \frac{n\zeta}{2}$$

$$= n^2 \left(e^{\frac{1}{2}(\lambda - i\mu)} - e^{-\frac{1}{2}(\lambda - i\mu)} \right)^2 \Big/ \left(e^{\frac{n}{2}(\lambda - i\mu)} - e^{-\frac{n}{2}(\lambda - i\mu)} \right)^2$$

As $z \to \pm c$, either $\lambda \to \infty$ or $\lambda \to -\infty$, and in either case one exponential becomes negligibly small, while in the other $-i\mu$ may be ignored. It thus appears that $dz'/dz \to \infty$ if $n < 1$, but $dz'/dz \to 0$ if $n > 1$.

The transformation has several uses; see Sections 80 and 89.

(See Reference: v. Kármán and Trefftz[77] and Müller[78].)

89. CIRCULAR-ARC CYLINDER, BOSS OR GROOVE

By means of the preceding transformation the flow can be found past any cylinder whose contour consists of two circular arcs. Only the symmetrical case will be treated here; compare Figure 142a and b.

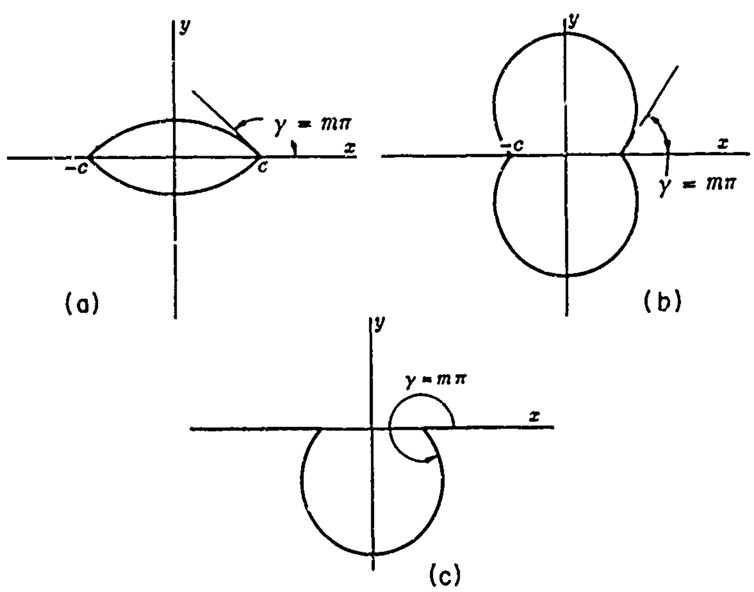


Figure 142 – Examples of a symmetrical circular-arc cylinder (a) or (b), or a circular-arc groove (c) in a plane wall.

To be streamlines, the arcs must transform into part of the real axis of w. Let the edges of the cylinder be at $(\pm c, 0)$ on the z-plane, so that the arcs have a common chord of length $2c$, and let each arc make a numerical angle $\gamma = m\pi$ with their common chord produced. Thus $0 < m < 1$, and the internal angle at each edge of the cylinder is $2(1 - m)\pi$; the radius of each arc is $R = c/\sin\gamma$. Then the transformation Equation [88a] flattens both arcs onto the same segment of the real axis of z' provided $n = 1/m$; the region outside of the cylinder thus goes into the whole z'-plane, and at infinity $z' \to z$.

Hence substituting also w/U for z', a complex potential w defined by

$$\frac{mw - cU}{mw + cU} = \left(\frac{z - c}{z + c}\right)^{\frac{1}{m}} \qquad [89a]$$

will represent flow past the given cylinder, with a uniform velocity U toward negative x at infinity. Then, in terms of ζ, λ, and μ as defined by Equations [88d, e, f, k] or [88l, m, n],

$$w = \frac{cU}{m} \coth \frac{\zeta}{2m} ; \qquad [89b]$$

$$\phi = \frac{cU}{2mG} \sinh \frac{\lambda}{m} , \quad \psi = \frac{cU}{2mG} \sin \frac{\mu}{m} , \qquad [89c, d]$$

$$G = \sinh^2 \frac{\lambda}{2m} + \sin^2 \frac{\mu}{2m} = \frac{1}{2} \left(\cosh \frac{\lambda}{m} - \cos \frac{\mu}{m}\right) ; \qquad [89e]$$

$$\frac{dw}{dz} = \frac{dw}{d\zeta} \left(\frac{dz}{d\zeta}\right)^{-1} = \frac{U}{m^2} \left(\sinh \frac{\zeta}{2} \Big/ \sinh \frac{\zeta}{2m}\right)^2 , \qquad [89f]$$

$$q = \left|\frac{dw}{dz}\right| = \frac{|U|}{m^2 G} \left(\sinh^2 \frac{\lambda}{2} + \sin^2 \frac{\mu}{2}\right) = \frac{|U|}{2m^2 G} (\cosh \lambda - \cos \mu). \qquad [89g]$$

On the x-axis where $|x| > c$ and $\mu = \theta_1 - \theta_2 = 0$, $\zeta = \lambda$, $z = x$, $q = |u|$ and

$$u = - \frac{U}{m^2} \frac{\cosh \lambda - 1}{\cosh (\lambda/m) - 1} , \quad \lambda = \ln \frac{x + c}{x - c} . \qquad [89h, i]$$

On the y-axis outside of the cylinder, $\lambda = 0$, $\zeta = - i\mu$, $q = |u|$ and

$$u = - \frac{U}{m^2} \frac{1 - \cos \mu}{1 - \cos (\mu/m)} , \quad \mu = 2 \cot^{-1} \frac{y}{c} . \qquad [89j, k]$$

On the cylinder itself $\mu = \pm m\pi$ and, from Equation [89g],

$$q = \frac{|U|}{m^2} \frac{\cosh \lambda - \cos (m\pi)}{\cosh (\lambda/m) + 1} . \qquad [89l]$$

The maximum velocity occurs at the middle of the sides, where $\lambda = 0$, and is

$$q_{max} = \frac{|U|}{2m^2}(1 - \cos m\pi). \qquad [89m]$$

The edges are stagnation lines; for, with $0 < m < 1$, $\cosh \lambda/\cosh(\lambda/m) \to 0$ as $\lambda \to \pm\infty$.

Half of the symmetrical streamlines past such a cylinder are illustrated in Figure 143 for $m = 0.88$, $\gamma = 158$ deg, and by the upper part of Figure 144 for $m = 0.71$, $\gamma = 127$ deg. In both cases the pressure in steady motion is shown as $p - p_\infty$, along the surface of the cylinder and the outlying parts of the plane of symmetry. The case $m = 1/3$, $\gamma = 60$ deg is illustrated in Figure 157, page 232. If $m = 1/2$, the cylinder is circular.

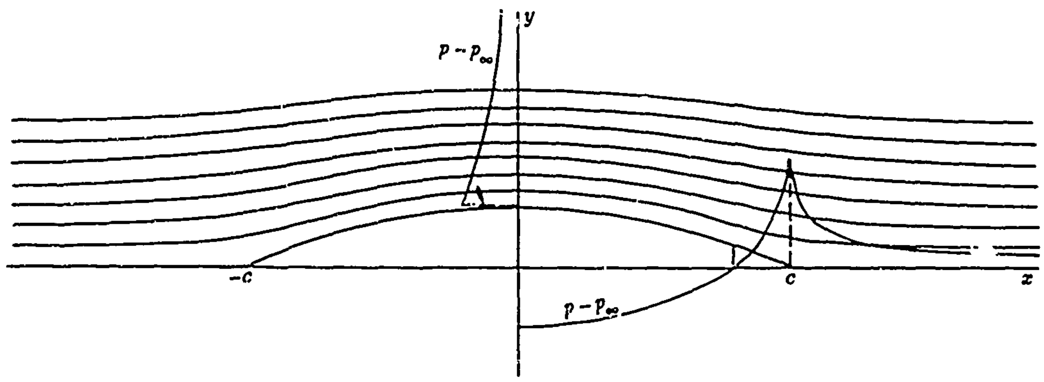


Figure 143 – Streamlines past a symmetrical circular-arc cylinder, and pressure p in steady flow, either on the cylinder or in the fluid along the x or y axis.

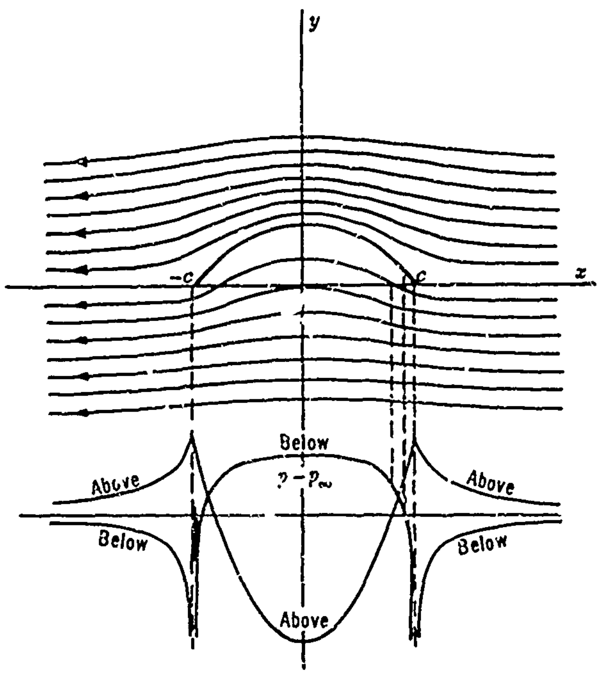


Figure 144 – Streamlines near a thin sheet containing the x axis except for a circular-arc buckle, and also the pressure p on both sides of the sheet and the buckle

The *motion* of such a cylinder through fluid stationary at infinity, at velocity U *parallel to the common chord* of the arcs, can be represented by adding a term $-Uz$ in w. Then, at points where ζ is small and hence z large, using Equation [33i] in Equations [88k] and [89b],

$$z = c\left(\frac{2}{\zeta} + \frac{\zeta}{6}\dots\right),$$

$$w = \frac{cU}{m}\left(\frac{2m}{\zeta} + \frac{\zeta}{6m}\dots\right) - Uz = \frac{cU}{6}\left(\frac{1}{m^2} - 1\right)\zeta\dots = \frac{c^2U}{3z}\left(\frac{1}{m^2} - 1\right)\dots$$

Thus, in Equations [76c, d, f], $(R)\ b_1 = b_1 = (c^2U/3)\,(1/m^2 - 1)$, and the kinetic energy of the fluid per unit length of the cylinder is

$$T_1 = \frac{1}{2}\rho\left[\frac{2\pi}{3}\left(\frac{1}{m^2} - 1\right)c^2 - S\right]U^2, \qquad [89n]$$

where S is the cross-sectional area of the cylinder or

$$S = \frac{c^2}{\sin^2 m\pi}\,[2(1 - m)\,\pi + \sin 2m\pi]. \qquad [89o]$$

The case of *motion perpendicular to the chord* is also easily treated by noting first that the slightly modified transformation $mz' = Z = c \coth(\zeta/2m)$, with $z = c\coth(\zeta/2)$ as before, flattens the outline of the cylinder into the segment of the real Z-axis from $Z = -c$ to $Z = c$. The cylinder is thereby transformed into a lamina of width $2c$. From Equation [85b] for transverse flow on the Z-plane past such a lamina, $w = -iU\,(Z^2 - c^2)^{1/2}$. Furthermore, at infinity, $Z \to 2mc/\zeta \to mz$, so that uniform flow on the Z-plane transforms into similar flow on the z-plane but with the velocity multiplied by m (since $w = UZ$ becomes $w = mUz$). Hence, after multiplying w by $1/m$ in order to keep the stream velocity equal to U on the z-plane, and substituting for Z in terms of z,

$$w = -\frac{icU}{m}\left(\sinh\frac{\zeta}{2m}\right)^{-1}. \qquad [89p]$$

This is the complex potential for flow past the original cylinder at velocity U toward negative y. Then

$$\phi = \frac{cU}{mG} \cosh \frac{\lambda}{2m} \sin \frac{\mu}{2m}, \quad \psi = -\frac{cU}{mG} \sinh \frac{\lambda}{2m} \cos \frac{\mu}{2m},$$

$$q = \left| \frac{dw}{d\zeta} \frac{d\zeta}{dz} \right| = \left| -\frac{iU}{m^2} \cosh \frac{\zeta}{2m} \sinh^{-2} \frac{\zeta}{2m} \sinh^2 \frac{\zeta}{2} \right|$$

$$= \frac{U}{2m^2 G} (\cosh \lambda - \cos \mu) \left[\frac{1}{2} \left(\cosh \frac{\lambda}{m} + \cos \frac{\mu}{m} \right) \right]^{½}. \quad [89q]$$

When the fluid velocity at infinity is again suppressed, this time by adding iUZ in w so as to superpose a flow toward positive y, then at large z and small ζ writing $z = 2c/\zeta + c\zeta/6 \dots$ again,

$$w = -\frac{icU}{m} \left(\frac{\zeta}{2m} - \frac{\zeta^3}{48m^3} \dots \right)^{-1} + iUz$$

$$= icU \left(\frac{1}{12m^2} + \frac{1}{6} \right) \zeta = i \frac{c^2 U}{3z} \left(\frac{1}{2m^2} + 1 \right) \dots \quad [89r]$$

Thus, from Equation [76d, e], with $\gamma = \pi/2$, $e^{-i\pi/2} = -i$,

$$T_1 = \frac{1}{2} \rho \left[\frac{2\pi}{3} \left(\frac{1}{2m^2} + 1 \right) c^2 - S \right] U^2. \quad [89s]$$

Cylindrical Boss or Groove. According to Equations [89d] and [88m], the x-axis for $|x| > c$ is part of a streamline; hence semi-infinite walls can be inserted there. Provided $m < 1$, half of the field then represents flow past a plane wall interrupted by a cylindrical boss of circular-arc section, which is 2c wide at the base and has an external angle γ or $m\pi$ between its tangent and the wall, and hence a radius $R = c/\sin m\pi$. Figures 143 and 157 and the upper part of Figure 144 may also be interpreted as showing streamlines for such a flow.

If $m > 1$, the diagram on the z-plane overlaps on itself and the whole field cannot be used. Provided $m \leqq 2$, however, the upper half of the w-plane taken by itself maps conformally upon the part of the z-plane that lies above the part of the x-axis on which $|x| > c$ and also above the arc $\mu = m\pi$, which now lies below the axis. For this purpose take $0 \leqq \theta_1 < 2\pi$, $-\pi < \theta_2 \leqq \pi$; then $0 \leqq \mu < 2\pi$ and below the x-axis μ lies in the third or fourth quadrant.

A wall can be inserted along the arc and the outlying parts of the x-axis on which $\psi = 0$, and the formulas then represent flow past a circular-arc groove in a plane wall; see Figure 142c. The velocity is infinite at the projecting edges of the groove.

The lower part of Figure 144 represents the flow past such a groove with $m = 1.30$, $\gamma = 233$ deg; to match the description, the figure needs to be turned upside down and the x-axis reversed. The entire figure may also represent flow on both sides, with the same velocity at infinity, past a thin sheet with a circular-arc buckle in it. The sheet cannot be removed, since the pressures are unequal on the two sides; the excess of pressure $p - p_\infty$ is shown in the figure for both sides, on an arbitrary scale, on the assumption of steady motion. Streamlines past a deeper groove, with $m = 1.75$, $\gamma = 315$ deg, are shown in Figure 145.

(For notation and method; see Section 34; Reference 2, Section 6.51, where $n = 2m$; J.L. Taylor,[33] where $m = 1/k$.)

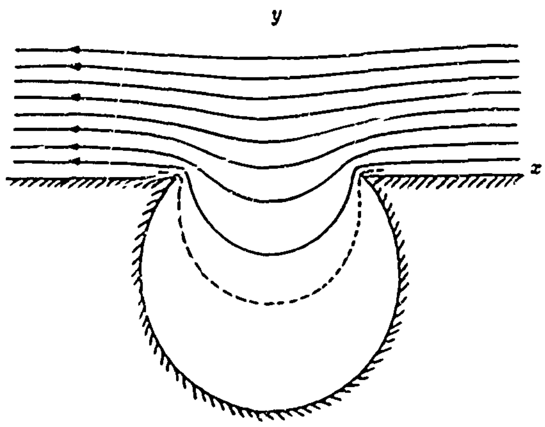


Figure 145 – Flow past a sheet or wall with a deep circular-arc groove.

90. DOUBLE CIRCULAR CYLINDER, OR CYLINDER AGAINST A WALL

If, in the formulas of the last section, m is made zero while c remains finite, both arcs come into coincidence with the outlying x-axis. By decreasing c as well, however, the arcs can be kept finite. Their radius is $R = c/\sin(m\pi)$, since each subtends an angle $2(\pi - m\pi)$ at its center; and R remains equal to a fixed number a if c is kept equal to $a \sin(m\pi)$ as $m \to 0$. The arcs thus reduce to two circles of radius a touching both each other and the x-axis at the origin.

Then, at a fixed point representing a given value of z, ζ becomes small in Equation [88k] as $c \to 0$, and $z \to 2c/\zeta$; substituting for ζ in Equation [89b] and noting that in the limit, as $m \to 0$, $c/m = a \sin(m\pi)/m \to a\pi$,

$$w = \pi a U \coth \frac{\pi a}{z} . \qquad [90a]$$

Hence, using hyperbolic formulas in Section 32 and $z = x + iy$,

$$\phi = \frac{\pi a U}{2G} \sinh \frac{2\pi a x}{r^2}, \quad \psi = -\frac{\pi a U}{2G} \sin \frac{2\pi a y}{r^2} \qquad [90b, c]$$

$$G = \frac{1}{2}\left(\cosh \frac{2\pi a x}{r^2} - \cos \frac{2\pi a y}{r^2}\right), \quad r = (x^2 + y^2)^{1/2}, \qquad [90d, e]$$

$$q = \left|\frac{dw}{dz}\right| = \frac{\pi^2 a^2}{r^2 G} |U|. \qquad [90f]$$

These formulas represent a uniform stream flowing past two cylinders of radius a in contact along a common generator, which passes through the origin. The fluid approaches at velocity U toward negative x and hence perpendicularly to the plane through the axes of the cylinders, as illustrated in Figure 146.

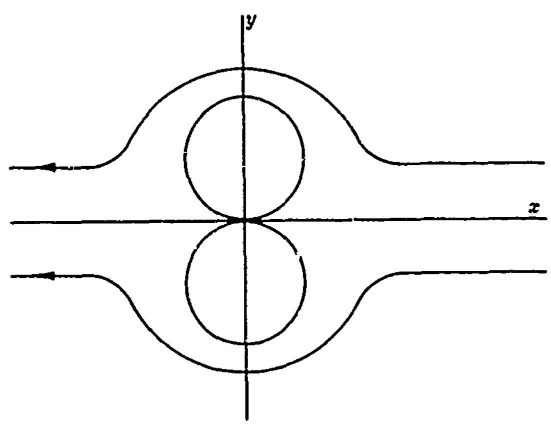


Figure 146 – Flow past two similar cylinders in contact along a common generator.

Or, if a boundary is inserted along the x-axis and only half of the diagram is used, the flow is represented past a cylinder resting against a plane wall. Streamlines for the latter case are shown in Figure 147. The excess of pressure above the pressure at infinity, for steady motion, is shown in the figure as $p - p_\infty$, along the positive x-axis up to the origin, and then around the right-hand half of the cylinder; the abscissa for the latter part of the curve represents the angle θ, plotted toward the left.

On the x-axis, $\psi = 0$, $r = \pm x$, $q = |u|$, and

$$u = -\frac{2\pi^2 a^2 U}{x^2}\left(\cosh \frac{2\pi a}{x} - 1\right)^{-1} = -\frac{\pi^2 a^2 U}{x^2}\left(\sinh \frac{\pi a}{x}\right)^{-2} \qquad [90g]$$

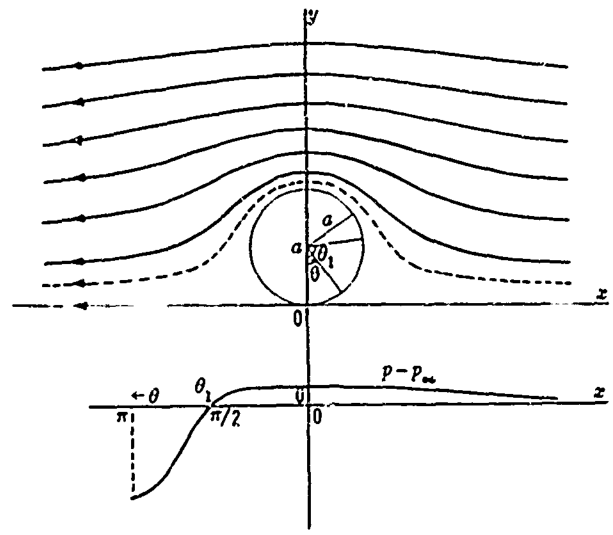


Figure 147 – Streamlines past a circular cylinder resting against a plane wall, also the pressure p during steady motion, along the wall for $x > 0$ and then around half of the circular surface.

Thus $u \to 0$ at $x = 0$, the sole stagnation point; for sinh Q increases much faster than Q as $x \to \infty$.

On the y-axis, $\phi = 0$, $r = \pm y$, $q = |u|$, and

$$u = -\frac{2\pi^2 a^2 U}{y^2}\left(1 - \cos\frac{2\pi a}{y}\right)^{-1} = -\frac{\pi^2 a^2 U}{y^2}\left(\sin\frac{\pi a}{y}\right)^{-2} \qquad [90h]$$

On the cylinders, $x^2 + (y \mp a)^2 = a^2$, or $r^2 = \pm 2ay$, hence $\psi = 0$ and

$$q = \frac{\pi^2 a|U|}{|y|}\left(\cosh\frac{\pi x}{y} + 1\right)^{-1} = \frac{\pi^2 |U|\, osc^2\, \dfrac{\theta}{2}}{2\left[1 + \cosh\left(\pi \cot\dfrac{\theta}{2}\right)\right]}, \qquad [90i]$$

in terms of an angle θ defined as in Figure 147. The maximum velocity, occurring at $x = 0$ and $|y| = 2a$, is $\pi^2|U|/4$ or $2.47\,|U|$.

Flow Parallel to the Line of Axes. Equation [89p] gives similarly in the limit, with U replaced by V,

$$w = -i\pi aV \left(\sinh \frac{\pi a}{z}\right)^{-1}, \qquad [90j]$$

for flow at infinity at velocity V toward negative y and hence parallel to the line of axes of the touching cylinders. Then

$$\phi = \frac{\pi aV}{G} \cosh \frac{\pi ax}{r^2} \sin \frac{\pi ay}{r^2}, \quad \psi = -\frac{\pi aV}{G} \sinh \frac{\pi ax}{r^2} \cos \frac{\pi ay}{r^2} \qquad [90k, l]$$

$$q = \frac{\pi^2 a^2 |V|}{r^2 G} \left[\frac{1}{2}\left(\cosh \frac{2\pi ax}{r^2} + \cos \frac{2\pi ay}{r^2}\right)\right]^{1/2}. \qquad [90m]$$

As the origin is approached along or between the cylinders, $\cosh (2\pi a\, x/r^2)$ increases without limit and $q \to 0$.

The *kinetic energy* T_1 of the fluid near unit length of the double cylinder, when moving at velocity U through fluid that is at rest at infinity, can be found conveniently by substituting $a\pi$ for c/m in Equations [89n] and [89s] and then letting $c \to 0$. This gives, since $S = 2\pi a^2$: for motion perpendicular to the line of axes,

$$T_1 = \frac{1}{2} \rho \left(\frac{\pi^2}{3} - 1\right) 2\pi a^2 U^2 = \frac{1}{2} \rho\, (4.580\, \pi a^2)\, U^2; \qquad [90n]$$

for motion parallel to the line of axes,

$$T_1 = \frac{1}{2} \rho \left(\frac{\pi^2}{6} - 1\right) 2\pi a^2 U^2 = \frac{1}{2} \rho\, (1.290\, \pi a^2)\, U^2. \qquad [90o]$$

In the case of motion perpendicular to the line of axes, a wall can be inserted, as before; then half of T_1 is the kinetic energy of the fluid near unit length of a cylinder that is sliding along a wall.

(For notation and method; see Section 34; Reference 2, Section 6.52.)

91. CYLINDERS OF OTHER FORMS

Aside from airfoil shapes, cylinders of the following additional cross-sectional shapes have been studied, the cylinder being either stationary in a stream or moving through quiescent fluid. The kinetic energy per unit length of fluid of density ρ surrounding the cylinder, when it is moving at velocity U with the fluid quiescent at infinity, is denoted below by T_1; see also Section 34.

(a) A hypocycloid, by Agostinelli[108] and Sestini.[109]

(b) Rectangular, by Riabouchinsky[110] and J.L. Taylor.[33] For kinetic energy, see Chapter V.

(c) Equal-sided quadrilateral of side s, moving parallel to a diagonal bisecting an internal angle of θ radians, by J.L. Taylor.[33] Here $z = \int [w^2/(w^2 - 1)]^{\theta/(2\pi)}\, dw$. The area is $s^2 \sin\theta$ and

$$T_1 = \frac{1}{2}\rho \left[\theta \left(\frac{2\Gamma(3/2)}{\Gamma\left(1 - \frac{\theta}{2\pi}\right)\Gamma\left(\frac{\theta}{2\pi} + \frac{1}{2}\right)} \right)^2 - \sin\theta \right] s^2 U^2 \qquad [91a]$$

where $\Gamma(x)$ denotes the gamma function of x.

(d) Two parabolic arcs meeting at right angles, by J.L. Taylor.[33] If h is the length of the chord joining the edges, the area is $h^2/3$, and

(1) For motion parallel to the chord

$$z = \frac{b}{4}\left[\int (w^2 - 1)^{-1/4}\, dw\right]^2, \quad T_1 = \frac{1}{2}\rho \frac{h^2}{3}\left(\frac{4K^4}{\pi^3} - 1\right) U^2 = \frac{1}{2}\rho(0.178)\, h^2 U^2; \qquad [91b,c]$$

(2) For motion perpendicular to the chord,

$$z = \frac{b}{4}\left[\int \left(\frac{w}{w^2 - 1}\right)^{1/2} dw\right]^2, \quad T_1 = \frac{1}{2}\rho \frac{h^2}{3}\left(\frac{8K^4}{\pi^3} - 1\right) U^2 = \frac{1}{2}(0.683)\, h^2 U^2. \qquad [91d,e]$$

Here b is a constant and K is the complete elliptic integral of modulus $\sqrt{1/2}$ or

$$K = \int_0^{\pi/2} \left(1 - \frac{1}{2}\sin^2\theta\right)^{-1/2} d\theta = 1.8541.$$

(e) Four equal semicircles, on the sides of an inscribed square whose diagonal is of length D, by J.L. Taylor.[33] The area is $(2 + \pi)\, D^2/4$ and

$$w = t + \frac{1}{t}\ , \ z = \left[\int_0^t (1 - t^4)^{-1/2}\, dt\right]^{-1},$$

$$T_1 = \frac{1}{2}\,\rho\,(\pi K^2 - \pi - 2)\,\frac{D^2}{4}\,U^2 = \frac{1}{2}\,\rho\,(1.414)\,D^2\,U^2, \qquad [91f]$$

where K is as in (d).

(f) Circle with radial plane extending to infinity on one side:

$$w = C\left(\sqrt{\frac{z}{a}} + \sqrt{\frac{a}{z}}\right) + U\left(z + \frac{a^2}{z}\right),\ C \text{ and } U \text{ real.}$$

A few streamlines for positive values of C and U are shown in Figure 148; see Cisotti.[111]

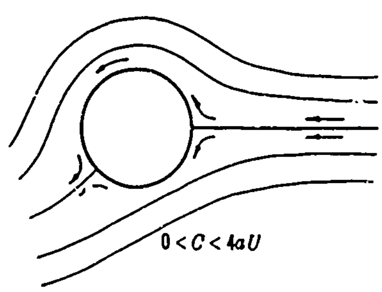


Figure 148a

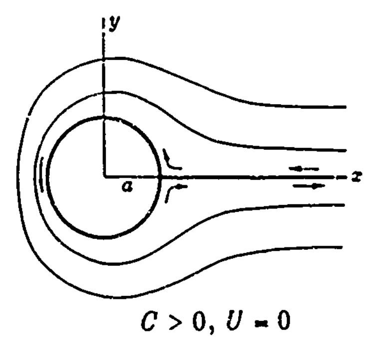


Figure 148b

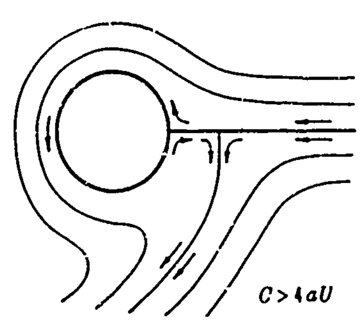


Figure 148c

Figure 148 – Streamlines around a circular cylinder attached to a semi-infinite plane. (See Reference 111.)

(g) Circle with a radial fin of finite width, by Bassani.[112]

(h) Circle with two equal radial fins opposite each other, by J.L. Taylor.[33] Let the greatest width between the outer edges of the fins be n times the diameter of the circle. Then, for motion perpendicular to the plane of the fins,

$$z = \frac{1}{2}\left(\sqrt{u^2 - k^2} + \sqrt{u^2 - 4 - k^2}\right), \quad k = n - \frac{1}{n},$$

and

$$T_1 = \frac{1}{2} \rho \pi (1 + k^2) a^2 U^2 \qquad [91g]$$

where a is the radius of the circle.

(i) Broken line, or two plane laminas joined by their edges, by Morton,[106] by Sona,[115] and, with special reference to an airfoil equipped with a rudder, by Sauer,[116] who gives other references. Streamlines for the flow without circulation past a right angle, approaching at 15 deg to the plane of one side, are shown in Figure 149. By properly choosing the circulation and the direction of approach of the stream, the velocity can be made finite at both edges of the lamina, whatever its angular aperture, as in Figure 150; see Sona.[115]

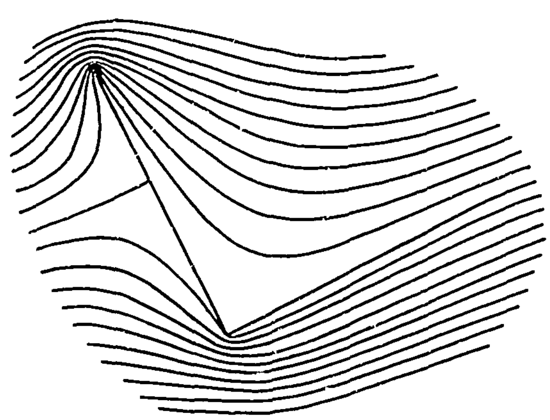


Figure 149 – Flow past a right-angle. See Section 91(i). (Copied from Reference 106.)

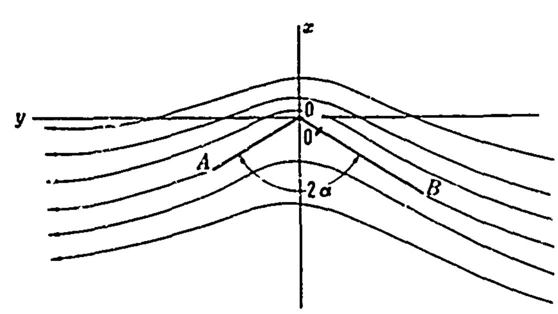


Figure 150 – Streamlines past an angle-lamina AB, with the direction of approach of the stream and the circulation around the lamina so chosen that the velocity is finite at both edges. See Section 91(i). (Copied from Reference 115, Lincei 22.)

(j) Rectangular cross, by Westwater.[117]

(k) Arc and line joined together, by Sauer.[116]

(l) A small circular cylinder of radius a in motion but momentarily coaxial with a surrounding stationary square cylindrical shell of width $2c$, by J.L. Taylor:[33]

$$T_1 = \frac{1}{2}\,\rho\pi a^2\left(1 + 1.719\,\frac{a^2}{c^2}\right). \qquad [91h]$$

(m) A convex curve joined to a straight segment, by Riabouchinsky;[110]

(n) Contours described parametrically by

$$x = a\cos u + (b\cos 2u)/2,\; y = b\sin u - (b\sin 2u)/2,$$

by Wrinch,[118] or by other analytical formulas, by Morris,[73, 119] Basset,[120] Neronoff,[121] Milne-Thompson.[122]

A general method of modifying the Schwarz-Christoffel transformation so as to introduce rounded edges or other curvatures, by adding "curve factors," was described with examples by Leathem;[74] see also page 123.

92. TWO EQUAL LINE DIPOLES WITH AXES LONGITUDINAL; FLOW PAST ONE OR TWO SIMILAR CYLINDERS

$$w = Uz + \mu\left(\frac{1}{z-b} + \frac{1}{z+b}\right), \qquad [92a]$$

U, μ and b real constants and $b > 0$,

$$\phi = Ux + \mu\left(\frac{\cos\theta_1}{r_1} + \frac{\cos\theta_2}{r_2}\right) = Ux + \mu\left(\frac{x-b}{r_1^2} + \frac{x-b}{r_2^2}\right), \qquad [92b]$$

$$\psi = Uy - \mu\left(\frac{\sin\theta_1}{r_1} + \frac{\sin\theta_2}{r_2}\right) = Uy - \mu y\left(\frac{1}{r_1^2} + \frac{1}{r_2^2}\right), \qquad [92c]$$

where $r_1 = [(x-b)^2 + y^2]^{1/2}$, $r_2 = [(x+b)^2 + y^2]^{1/2}$, and the significance of the angles θ_1 and θ_2 is shown in Figure 151. Hence

$$u = -U + \mu\left(\frac{\cos 2\theta_1}{r_1^2} + \frac{\cos 2\theta_2}{r_2^2}\right), \quad v = \mu\left(\frac{\sin 2\theta_1}{r_1^2} + \frac{\sin 2\theta_1}{r_2^2}\right), \qquad [92d, e]$$

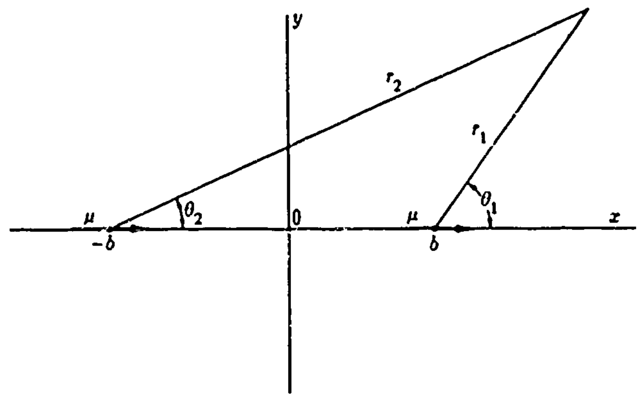


Figure 151 – Two equal line dipoles with longitudinal axes (in the direction of x).

$$q^2 = U^2 - 2\mu U \left(\frac{\cos 2\,\theta_1}{r_1^2} + \frac{\cos 2\,\theta_2}{r_2^2}\right) + \mu^2 \left(\frac{1}{r_1^4} + \frac{1}{r_2^4} + \frac{2\cos(2\theta_1 - 2\theta_2)}{r_1^2\, r_2^2}\right) . \qquad [92f]$$

On the x-axis $u = u_1$ and on the y-axis $u = u_2$ where

$$u_1 = -U + 2\mu \frac{x^2 + b^2}{(x^2 - b^2)^2}, \quad u_2 = -U + 2\mu \frac{b^2 - y^2}{(b^2 + y^2)^2} . \qquad [92g, h]$$

A. *Two Line Dipoles Alone.* If $U = 0$, the field of flow is that due to two equal line dipoles whose axes are parallel and directed along the line through the locations of the dipoles, which are at $(\pm\, b, 0)$; compare Equation [37a]. Stagnation points occur on the y-axis at $(0, \pm\, b)$, where $\theta_1 = 135$ deg, $\theta_2 = 45$ deg, and $\psi = \mp\, \mu/b$. Streamlines for $|\psi| < |\mu|/b$ run through both dipoles; those for $|\psi| > |\mu|/b$ consist of two disconnected loops, one associated with each dipole. Streamlines above the x-axis are shown in Figure 152.

B. *Flow Past One or Two Similar Cylinders of Special Shape.* Assume that $U/\mu > 0$, so that the dipole axes are oppositely directed to the stream at infinity, whose velocity is U toward negative x. Then the streamline for $\psi = 0$ consists of the x-axis and the curve S defined by

$$\frac{1}{r_1^2} + \frac{1}{r_2^2} = \frac{2}{c^2}, \quad c = \sqrt{\frac{2\mu}{U}} . \qquad [92g, h]$$

If c is large, S is an oval curve cutting the x-axis in two stagnation points; as $c \to \infty$, it approximates a circle. If $c = b$, it contracts in the middle to a point at the origin; for $c < b$, it consists of two loops, one surrounding each dipole. Several possible forms of S are shown in Figure 153.

The formulas may represent flow past a cylinder, or two parallel similar cylinders, inserted along S. If $c < b/2$, the cross-sections of the cylinders are nearly circular; even if their

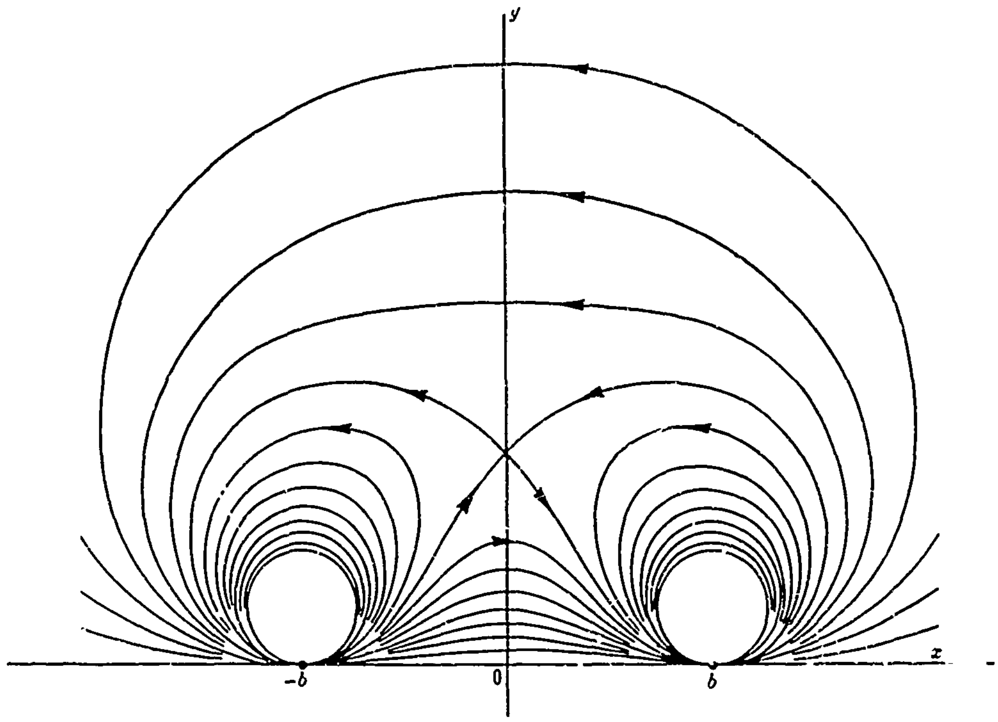


Figure 152 – Streamlines on one side of the plane of symmetry due to two line dipoles at $\pm b$. See Section 92. (Copied from Reference 124.)

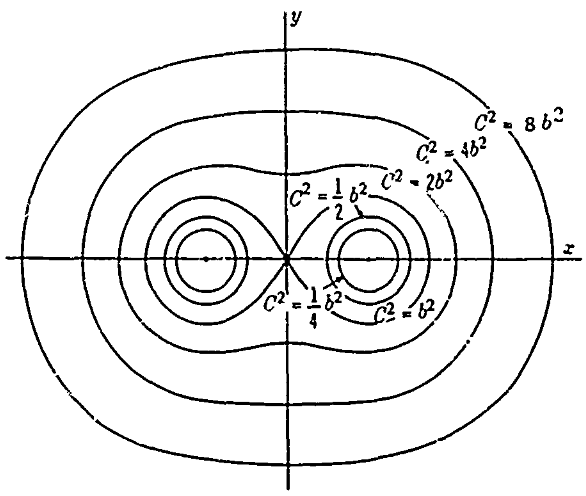


Figure 153 – Some possible forms for the dividing surface S due to two line dipoles in a stream. See Section 92. (Copied from Reference 124.)

diameters are as large as the distance between their surfaces, the maximum and minimum diameters of each cylinder differ by less than 4 percent; if their diameters are at most half of the distance between them, the difference is less than 1 percent.

In the case of two cylinders, the forces on them, which are equal and opposite, are easily found from the Blasius theorem. For the cylinder surrounding $z = b$, write du/dz thus:

$$\frac{du}{dz} = U - \mu \left[\frac{1}{(z-b)^2} + \frac{1}{4b^2} - \frac{(z-b)(z+3b)}{4b^2 (z+b)^2} \right].$$

Substitution in Equation [74g] and evaluation of the integral from the residues, as in Section 30, then gives for the force $X = \pi \rho \mu^2/(2b^3)$. The cylinders repel each other, because of lower velocities between them. For the approximately circular cylinder, $1/r_2^2$ may be dropped in comparison with $1/r_1^2$, and Equation [92g] then gives for its radius $a = r_1 = c/\sqrt{2} = \sqrt{\mu/U}$. Thus in this case the force is, approximately, $X = \pi \rho U^2 a^4/(2b^3)$.

(For notation and method; see Section 34; Reference Muller 124.)

93. TWO EQUAL LINE DIPOLES WITH AXES TRANSVERSE; FLOW PAST ONE OR BETWEEN TWO SIMILAR CYLINDERS.

$$w = Uz + \mu \left(\frac{1}{z - ib} + \frac{1}{z + ib} \right), \qquad [93a]$$

U, μ and b real constants and $b > 0$;

$$\phi = Ux + \mu \left(\frac{\cos \theta_1}{r_1} + \frac{\cos \theta_2}{r_2} \right) = Ux + \mu x \left(\frac{1}{r_1^2} + \frac{1}{r_2^2} \right), \qquad [93b]$$

$$\psi = Uy - \mu \left(\frac{\sin \theta_1}{r_1} + \frac{\sin \theta_2}{r_2} \right) = y \left[U - \mu \left(\frac{1}{r_1^2} + \frac{1}{r_2^2} - \frac{4b^2}{r_1^2 r_2^2} \right) \right], \qquad [93c]$$

where

$$r_1 = [x^2 + (y-b)^2]^{1/2}, \; r_2 = [x^2 + (y+b)^2]^{1/2}, \qquad [93d, e]$$

and the significance of θ_1 and θ_2 is shown in Figure 154. Hence

$$u = -U + \mu \left(\frac{\cos 2\theta_1}{r_1^2} + \frac{\cos 2\theta_2}{r_2^2} \right), \; v = \mu \left(\frac{\sin 2\theta_1}{r_1^2} + \frac{\sin 2\theta_2}{r_2^2} \right). \qquad [93f, g]$$

$$q^2 = U^2 - 2\mu U \left(\frac{\cos 2\theta_1}{r_1^2} + \frac{\cos 2\theta_2}{r_2^2} \right) + \mu^2 \left(\frac{1}{r_1^4} + \frac{1}{r_2^4} + \frac{2\cos(2\theta_1 - 2\theta_2)}{r_1^2 r_2^2} \right)$$

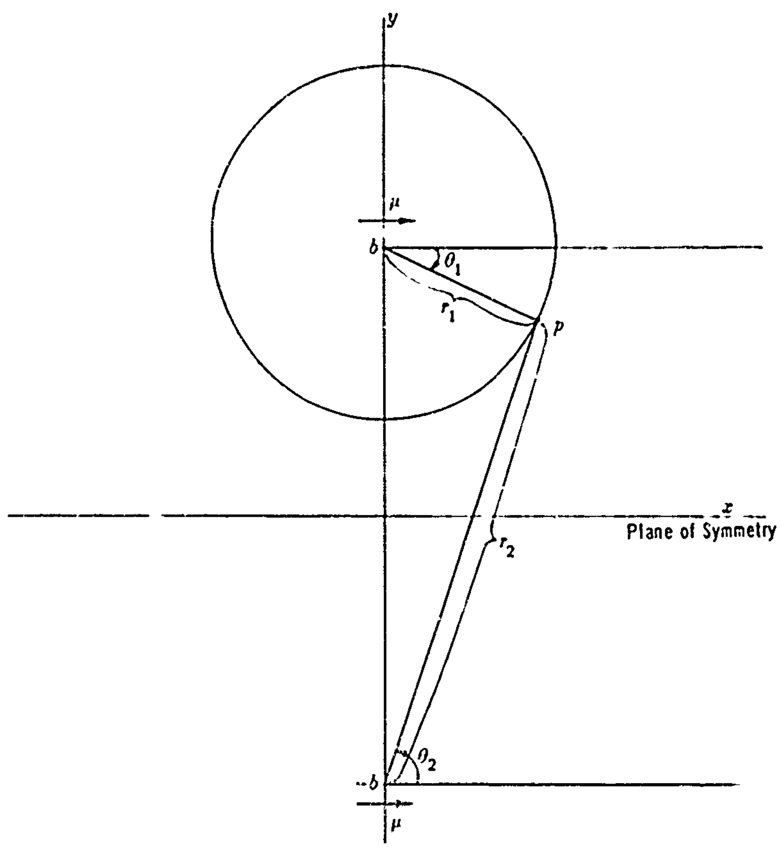


Figure 154 – Two equal line dipoles at (0, $\pm b$), with transverse axes (parallel to x). For the near-circle see Section 93B.

On the x-axis $u = u_1$ and on the y-axis $u = u_2$ where

$$u_1 = - U + 2\mu \frac{x^2 - b^2}{(x^2 + b^2)^2} \ , \ u_2 = - U - 2\mu \frac{y^2 + b^2}{(y^2 - b^2)^2} \ . \qquad [93h, i]$$

There is flow symmetry about the plane $y = 0$, mere geometrical symmetry about $x = 0$.

A. *Two Line Dipoles Alone.* If $U = 0$, the flow is that due to two equal line dipoles whose axes are parallel but directed at right angles to the line joining the locations of the dipoles, which are at (0, $\pm$ b); the dipole axes have the direction of the x-axis. Stagnation points occur on the x-axis at ($\pm$ b, 0); streamlines run to these points, divide, and continue in opposite directions along the x-axis. Some streamlines are shown, above the x-axis only, in Figure 155.

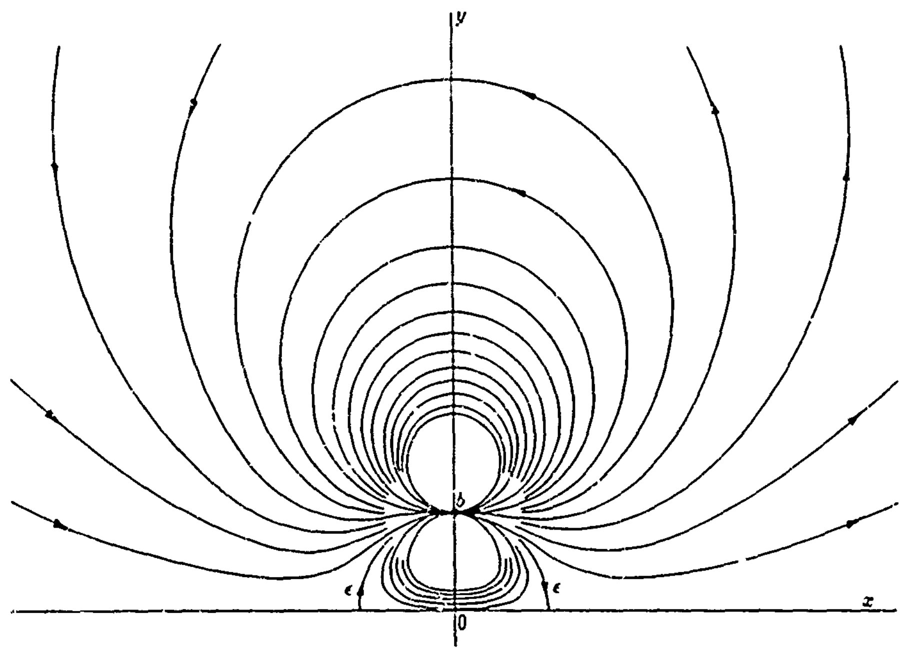


Figure 155 – Streamlines, above the plane of symmetry, due to two equal line dipoles with transverse axes. See Section 93(A). (Copied from Reference 124.)

B. *Flow Past One or Between and Around Two Similar Cylinders of Special Shape.* Assume $U/\mu > 0$, so that the dipole axes are oppositely directed to the stream at infinity, whose velocity is U toward negative x.

The streamline for $\psi = 0$ consists of the x-axis and also of the curve S defined by

$$\frac{1}{r_1^2} + \frac{1}{r_2^2} - \frac{4b^2}{r_1^2 r_2^2} = \frac{2}{c^2}, \quad c = \sqrt{\frac{2\mu}{U}}, \qquad [93j, k]$$

provided $c^2 \geqq 8b^2$. If c is large, S approximates a circle enclosing both dipoles. As c decreases, S becomes compressed along the x-axis; when $c^2 = 8b^2$, S consists of two circular arcs defined respectively by $r_1 = 2b$ and $r_2 = 2b$ and meeting at the stagnation points, which are then at $(\pm\sqrt{3}\,b, 0)$. This is a special case, for $n = 2/3$, of the flow considered in Section 89. As c^2 becomes less than $8b^2$, the curve defined by 93(j) disappears, and the dividing surface splits along the x-axis to form two separate curves, each of which surrounds one dipole and carries two stagnation points. The curves can be found by setting $u = v = 0$ in 93(f, g) in order to find the stagnation points and then calculating the value of ψ at these points from 93(c); with this constant value of ψ, 93(c) defines the curves. They soon approximate to circles, whose radius, for small c/b, approximates $c/\sqrt{2} = \sqrt{\mu/U}$.

The dividing surface may represent a cylinder, or two cylinders, of a certain shape, immersed in a uniform stream. The limiting form of S for $c^2 = 8b^2$ and a larger oval are shown in Figure 156. Half of S for the case $c = 0.94\,b$ is shown by the approximate circle in Figure 154. The streamlines for flow past the limiting form are shown, above the x-axis only, in Figure 157. If $c^2/8b^2$ is rather small, there is an approximation to two circular cylinders in a stream perpendicular to the line joining their centers, which are $2b$ apart.

The forces on the two cylinders can be found from the Blasius theorem, as in Section 92. The cylinders attract each other, because of higher velocities between them, with a force $F = \pi \rho \mu^2/(2b^3)$ acting on each. For slender circular cylinders of radius a, $F = \pi \rho U^2 a^4/(2b^3)$, approximately.

A rigid wall may also be inserted along the x-axis. Then half of the field represents flow past a cylinder of a certain shape with its axis distant b from a rigid wall. The force F, now directed toward the wall, remains the same.

(For notation and method; see Section 34; Reference Müller 124.)

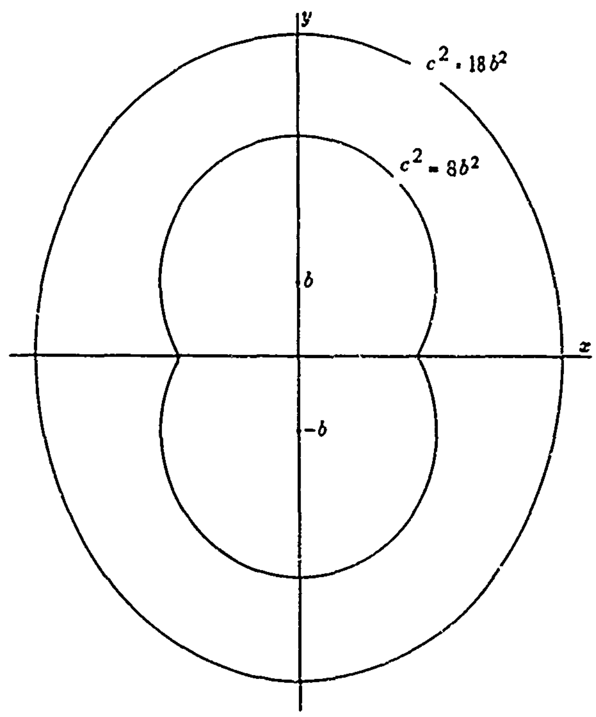


Figure 156 – Two possible forms of the dividing surface S for two equal line dipoles at $(0, \pm b)$ in a stream. See Section 93(B). (Copied from Reference 124.)

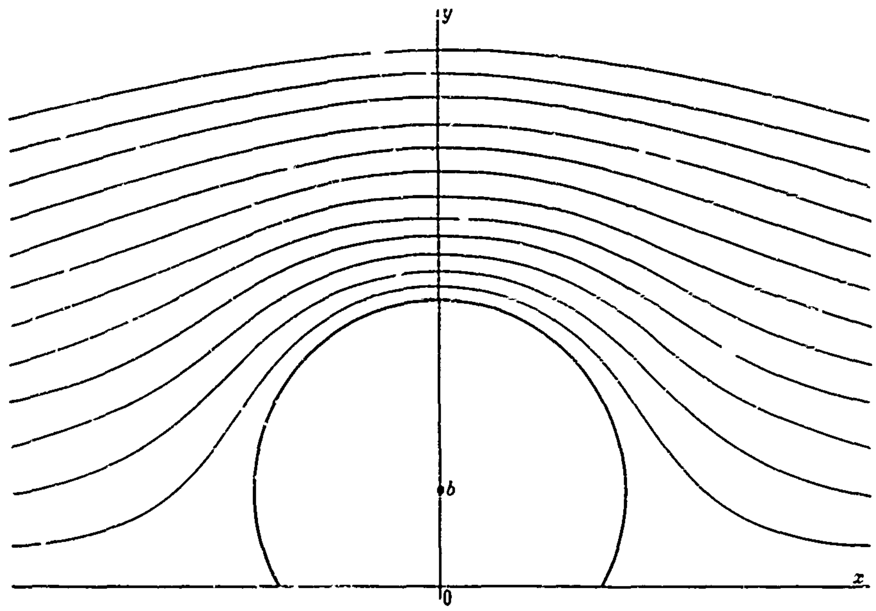


Figure 157 – Streamlines above the plane of symmetry, outside the dividing surface S obtained from two line dipoles in a stream. See Section 93(B). The boundary may also represent a circular boss on a plane wall, with the x-axis taken along the wall, or half of a symmetrical circular-arc cylinder, as in Section 89. (Copied from Reference 124.)

94. TWO CIRCULAR CYLINDERS IN A STREAM; CYLINDER AND WALL

Since a cylinder immersed in a uniform stream merely adds the flow due to a certain dipole on its axis, as in Section 67, and the image of a dipole in a circular cylinder is another dipole, as in Section 52, the flow around any number of cylinders in a stream can be built up in terms of an infinite train of image dipoles in each cylinder. Circulation around the cylinders may be added by assuming a suitable vortex on the axis of each, and then an infinite train of image vortices inside each, in accord with results in Section 42.

Only the first approximation to the solution will be given in detail here.

Let two cylinders A and B have radii a and b, and let their axes be located, respectively, at (0, 0) and $(-d, 0)$ so that they are d apart, and let a/d and b/d be small. Let the stream approach at an angle α with a line drawn through their axes, as illustrated in Figure 158; and let there be circulation Γ_1 about A and Γ_2 about B. Then a first approximation to the complex potential is

$$w = U\left[ze^{-i\alpha} + e^{i\alpha}\left(\frac{a^2}{z} + \frac{b^2}{z+d}\right)\right] + \frac{i}{2\pi}\left[\Gamma_1 \ln z + \Gamma_2 \ln (z+d)\right]. \quad [94a]$$

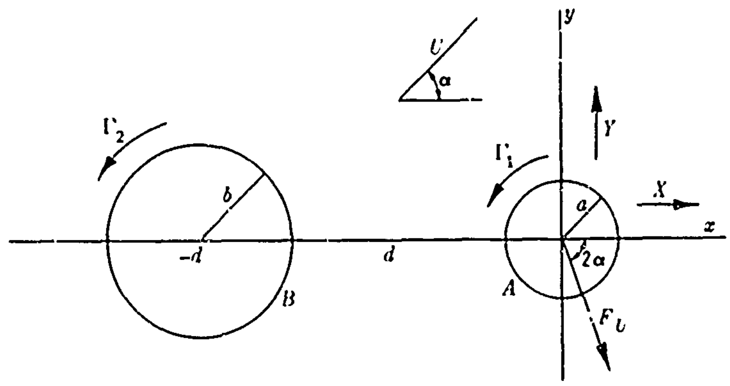


Figure 158 – Two circular cylinders in a stream. See Section 94.

The first term represents the flow past each cylinder as if the other were absent; it is constructed out of Equation [69j], in which, for the term representing the contribution of the second cylinder, z is replaced by $z + d$ in order to displace its axis to $(-d, 0)$.

The forces on the cylinders may be found from the Blasius theorem. Here

$$\frac{dw}{dz} = Ue^{-i\alpha} - Ue^{i\alpha}\left(\frac{a^2}{z^2} + \frac{b^2}{(z+d)^2}\right) + \frac{i}{2\pi}\left(\frac{\Gamma_1}{z} + \frac{\Gamma_2}{z+d}\right).$$

By proceeding as in Section 42, it is found from Equation [74g] that the force on A has components

$$X_1 = -\rho\Gamma_1 U \sin\alpha + \rho U \frac{a^2\Gamma_2 - b^2\Gamma_1}{d^2} \sin\alpha + \frac{\rho\Gamma_1\Gamma_2}{2\pi d} + 4\pi\rho U^2 \frac{a^2 b^2}{d^3} \cos 2\alpha, \quad [94b]$$

$$Y_1 = \rho\Gamma_1 U \cos\alpha + \rho U \frac{a^2\Gamma_2 - b^2\Gamma_1}{d^2} \cos\alpha - 4\pi\rho U^2 \frac{a^2 b^2}{d^3} \sin 2\alpha. \quad [94c]$$

The force X_2, Y_2 on B can be found by substituting in these expressions $-d$ for d and also interchanging a^2 with b^2 and Γ_1 with Γ_2. All terms containing d thus merely change sign.

The first terms of X and Y represent the usual lift, and the next terms a possible modification of the lift due to the presence of the other cylinder. The term in $\Gamma_1\,\Gamma_2$ represents a first approximation to the circulatory interaction, as found for a simpler case in Equation [42c'] of Section 42. It indicates that like circulations cause repulsion, unlike, attraction. The terms not containing a^2 or b^2 have the same values for two slender cylinders of any cross-sectional shapes.

There remains in addition a force of magnitude $4\pi\rho U^2 a^2 b^2/d^3$ inclined at an angle 2α to the line of axes. This force is shown as F_U in Figure 158. Its direction is such that a stream flowing parallel to the line of axes causes the cylinders to repel each other, whereas a transverse stream causes them to attract. When the stream is oblique the forces tend to turn the line of axes into a direction perpendicular to the stream.

If the stream is abolished by giving to everything an equal and opposite velocity, the forces remain the same. Then the cylinders, if free from circulation, repel each other when moving at equal speeds along the line of axes, but attract when moving in the same direction transversely to this axis.

It will be noted that no images at all have been included in this approximation, except those associated with the stream itself. Their inclusion leads to terms of higher order in $1/d$, for each of the three types of terms containing U^2, ΓU, or $\Gamma_1 \Gamma_2$, respectively.

If more than two slender cylinders are present, it is easily seen that the forces are additive in first approximation; the presence of each cylinder modifies the lift on every other and contributes interaction forces upon it according to the formulas just found.

The *complete solution* for noncirculatory flow past two cylinders of *equal radius* was written out in terms of images by Müller.[124] Streamlines for $\Gamma = 0$, and for $\alpha = 0$ and $\alpha = 90$ deg, respectively, are shown in Figure 159. The directions of the forces on the cylinders are also shown.

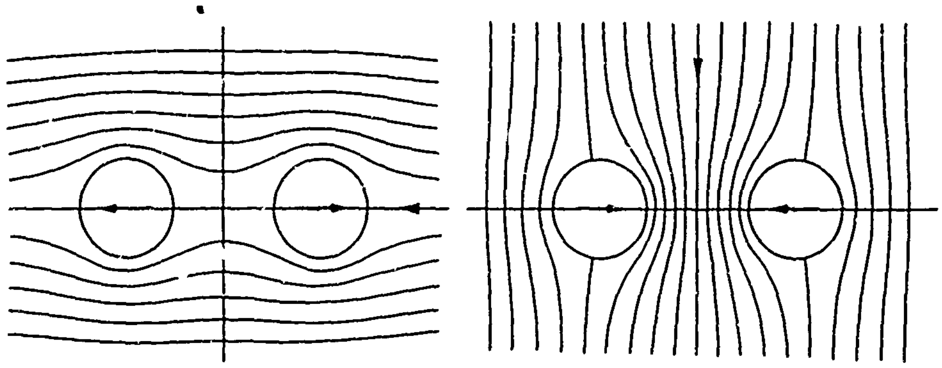


Figure 159 – Streamlines for flow in two directions past two equal circular cylinders. The arrows indicate the directions of the forces on the cylinders, when the motion is steady. (Copied from Reference 124.)

The flow was treated with no restriction upon the sizes of the cylinders except that they are external to each other, in terms of elliptic functions by Lagally,[125] and with use of bipolar coordinates and series by Endo.[126]

Cylinder and a Wall. If the two cylinders just considered have equal radii and equal and opposite circulations, or no circulations at all, and if the stream approaches perpendicularly to the line of axes, then the plane that bisects the line joining the axes is a stream surface and may be replaced by a rigid wall. Then half of the field represents flow along a wall and a slender circular cylinder parallel to it. For an exact solution, one of the treatments of two cylinders may be used, or an earlier direct treatment by Riabouchinsky.[127]

A first approximation to the force on the cylinder, which is equal and opposite to that on the wall, is given by Equation [94b, c].

Put $\alpha = 90$ deg, so that the x-axis is perpendicular to the wall, $\Gamma_1 = -\Gamma_2 = \Gamma$ where Γ is the circulation around the cylinder, and $d = 2h$ where h is the distance from the wall to the axis of the cylinder. Then $Y_1 = 0$ and, writing V instead of U for the velocity of the stream and X for the force on the cylinder,

$$X = -\rho\Gamma V\left(1 + \frac{a^2}{2h^2}\right) - \frac{\rho\Gamma^2}{4\pi h} - \frac{1}{2}\pi\rho V^2\frac{a^4}{h^3}, \qquad [94d]$$

where a is the radius of the cylinder. Thus the effect of V alone, or of Γ alone, is an attraction between the wall and the cylinder. The joint effect of circulation and stream is likewise an attraction when the two resulting components of velocity are in the same direction along the wall.

95. SLENDER CIRCULAR CYLINDERS MOVING INDEPENDENTLY, OR NEAR A WALL

When cylinders such as those considered in the last section move independently, the method of images can still be used, since a moving cylinder is equivalent to a dipole located on its axis, but the motion cannot be made steady by a suitable choice of the frame of reference. The forces can then be determined either by integration of the pressure or, more conveniently, by first finding the energy and then using the Lagrange equations. The latter method will be used here, to a first approximation only.

(A) Two Slender Cylinders

Let two parallel cylinders A and B have radii, respectively, a and b, and let A be moving at velocity V in a direction inclined at an angle α to a line PQ drawn through the axes of the cylinders in the direction from A toward B, while B is moving with velocity W inclined at the angle β to the same line; see Figure 160. Assume that there is no circulation around either cylinder; and for the present let the fluid be at rest at infinity.

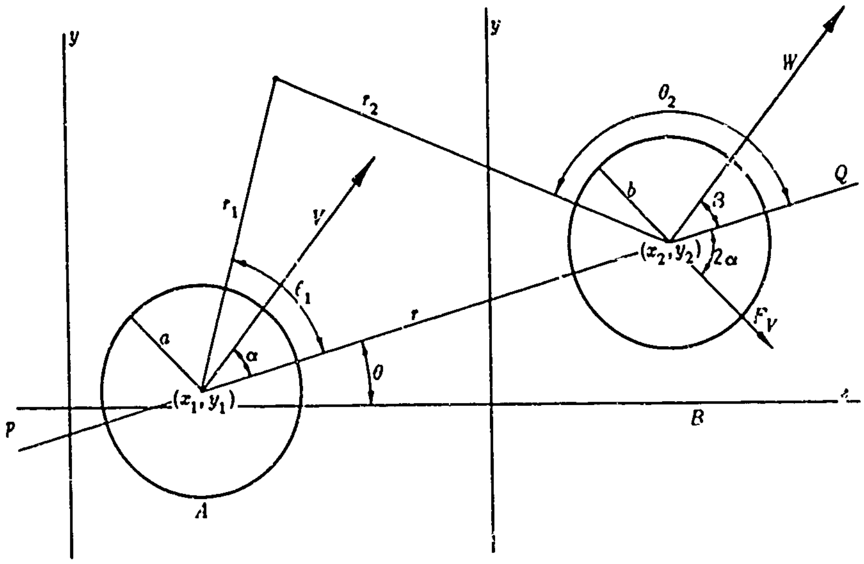


Figure 160 – Two slender circular cylinders in motion. See Section 95.

The first approximation to the velocity potential represents a dipole on the axis of each cylinder and is, from Equation [37s] in Section 37,

$$\phi' = a^2 V \frac{\cos(\theta_1 - \alpha)}{r_1} + b^2 W \frac{\cos(\theta_2 - \beta)}{r_2} \qquad [95a]$$

where r_1, r_2, θ_1, θ_2 are adequately defined in Figure 160. The flow due to each of these dipoles then violates the boundary condition at the surface of the other cylinder, hence their respective images in the other cylinder must be added, then, for a similar reason, the images of these images, and so on. Only the first pair of images will be included here.

Furthermore, to the same degree of approximation the displacement of the first images from the axis of the cylinder containing them may be neglected. Their contribution to the potential is then, from Equation [52c],

$$\phi'' = -\frac{a^2 b^2}{r^2} V \frac{\cos(\theta_2 + \alpha)}{r_2} - \frac{a^2 b^2}{r^2} W \frac{\cos(\theta_1 + \beta)}{r_1} \qquad [95b]$$

where r is the distance between the axes of the cylinders.

The contribution of cylinder B to the integral in Equation [17d], which expresses the kinetic energy of the fluid, is then

$$\frac{1}{2}\rho \int_0^{2\pi} \phi q_n \, ds = \frac{1}{2}\rho \int_0^{2\pi} (\phi' + \phi'') W \cos(\theta_2 - \beta)\, b\, d\theta_2 .$$

On the surface of B, r_2 equals b. Furthermore, to include only terms of order $1/r^2$, it is sufficiently accurate to write, by projection of the radius of B,

$$r_1 = r + b\cos\theta_2,\ \frac{1}{r_1} = \frac{1}{r} - \frac{b\cos\theta_2}{r^2},\ \sin\theta_1 = \frac{b\sin\theta_2}{r},\ \cos\theta_1 = 1;$$

and the second term of ϕ'' may be omitted.

The result of carrying out the integration, after adding and evaluating the corresponding integral over A, is the following simple expression for the total kinetic energy of the fluid per unit length of the cylinders:

$$T_1 = \frac{\pi}{2}\rho\left[a^2V^2 + b^2W^2 - \frac{4a^2b^2}{r^2}VW\cos(\alpha+\beta)\right]. \qquad [95c]$$

The next step is to express the energy in terms of coordinates representing the positions of the cylinders. Let the Cartesian coordinates of the axes of A and B be x_1, y_1, and x_2, y_2, and let the corresponding components of velocity be denoted by $\dot{x}_1$, $\dot{y}_1$, $\dot{x}_2$, $\dot{y}_2$. Let the line of axes PQ make an angle θ with the positive x-axis. Then, by projection,

$$V\cos\alpha = \dot{x}_1\cos\theta + \dot{y}_1\sin\theta,\ W\cos\beta = \dot{x}_2\cos\theta + \dot{y}_2\sin\theta,$$

$$V\sin\alpha = -\dot{x}_1\sin\theta + \dot{y}_1\cos\theta,\ W\sin\beta = -\dot{x}_2\sin\theta + \dot{y}_2\cos\theta.$$

Substitution for V and W in Equation [95c] gives

$$T_1 = \frac{\pi}{2}\rho\left\{a^2(\dot{x}_1^2 + \dot{y}_1^2) + b^2(\dot{x}_2^2 + \dot{y}_2^2)\right.$$

$$\left. - \frac{4a^2b^2}{r^2}\left[(\dot{x}_1\dot{x}_2 - \dot{y}_1\dot{y}_2)\cos 2\theta + (\dot{x}_1\dot{y}_2 + \dot{x}_2\dot{y}_1)\sin 2\theta\right]\right\}. \qquad [95d]$$

Here r and θ are to be considered as functions of x_1, y_1, x_2, y_2.

Now let X_{1B}, Y_{1B} denote components of the force exerted by the fluid on unit length of cylinder B. Then the reactive force on the fluid is $-X_{1B}$, $-Y_{1B}$, and the Lagrange equation for x_2 is

$$\frac{d}{dt}\frac{\partial T_1}{\partial \dot{x}_2} - \frac{\partial T_1}{\partial x_2} = -X_{1B},$$

where t denotes the time. Hence

$$- X_{1B} = \pi\rho \left\{ \frac{d}{dt} \left[b^2 \dot{x}_2 - \frac{2a^2 b^2}{r^2} (\dot{x}_1 \cos 2\theta + \dot{y}_1 \sin 2\theta) \right] \right.$$

$$- \frac{4a^2 b^2}{r^3} [(\dot{x}_1 \dot{x}_2 - \dot{y}_1 \dot{y}_2) \cos 2\theta + (\dot{x}_1 \dot{y}_2 + \dot{x}_2 \dot{y}_1) \sin 2\theta] \frac{\partial r}{\partial x_2}$$

$$\left. - \frac{4a^2 b^2}{r^2} [(\dot{x}_1 \dot{x}_2 - \dot{y}_1 \dot{y}_2) \sin 2\theta - (\dot{x}_1 \dot{y}_2 + \dot{x}_2 \dot{y}_1) \cos 2\theta] \frac{\partial \theta}{\partial x_2} \right\} .$$

A similar equation is obtained for Y_{1B}.

For simplicity, after differentiating with respect to t, let the axes be rotated so that at the instant under consideration $\theta = 0$. Then it is easily seen that at this instant

$$\frac{dr}{dt} = \dot{x}_2 - \dot{x}_1, \quad \frac{d\theta}{dt} = \frac{\dot{y}_2 - \dot{y}_1}{r}, \quad \frac{\partial r}{\partial x_2} = 1, \quad \frac{\partial r}{\partial y_2} = 0, \quad \frac{\partial \theta}{\partial x_2} = 0, \quad \frac{\partial \theta}{\partial y_2} = \frac{1}{r} .$$

Hence, with axes chosen so that $\theta = 0$, the force upon B has components

$$X_{1B} = \pi\rho \left[-b^2 \ddot{x}_2 + \frac{2a^2 b^2}{r^2} \ddot{x}_1 + \frac{4a^2 b^2}{r^3} (\dot{x}_1^2 - \dot{y}_1^2) \right] . \qquad [95e]$$

$$Y_{1B} = \pi\rho \left[-b^2 \ddot{y}_2 - \frac{2a^2 b^2}{r^2} \ddot{y}_1 - \frac{8a^2 b^2}{r^3} \dot{x}_1 \dot{y}_1 \right] , \qquad [95f]$$

where two dots denote two differentiations with respect to the time, or a component of acceleration. Here $\dot{x}_1 = V \cos\alpha$, $\dot{y}_1 = V \sin\alpha$.

Similar expressions are obtainable for the force on A, but it is also possible to use the equations just found by interchanging notation between the cylinders.

A stream at infinity can be introduced by changing to a uniformly moving frame of reference. This change does not affect the forces or the accelerations; the equations for X_{1B} and Y_{1B} as they stand can, therefore, be used in all cases, with the understanding that $\dot{x}_1$, $\dot{y}_1$, V and α all refer to the velocity of cylinder A taken relative to the fluid at infinity.

The first term in X_{1B} or Y_{1B} represents the usual effective increase in the inertia of B due to the presence of the fluid. The remaining terms represent the effect of A upon B, and are valid only if a/r and b/r are small. It is readily seen that the errors due to the approximations are of an order in $1/r$ that is higher than $1/r^2$ in terms that involve the acceleration and higher than $1/r^3$ in those that do not.

The interaction effect may be stated as follows, in first approximation:

(1) Acceleration of either cylinder evokes a force on the other whose magnitude equals $2\pi\rho a^2 b^2/r^2$ multiplied by the acceleration, and whose direction is inclined to a line drawn through the axes of the cylinders at the same angle as is the acceleration but on the opposite side of that line; compare Figure 161. The force and acceleration have the same direction when they lie along the line of axes, but opposite when they are transverse. For, $\ddot{x}_1$ and $\ddot{y}_1$ are the components of the acceleration of A, and the ratio of the terms containing them in X_{1B} and Y_{1B} gives $-\ddot{y}_1/\ddot{x}_1$ as the tangent of the inclination of the force to the x-axis, which is now assumed to be the line of axes.

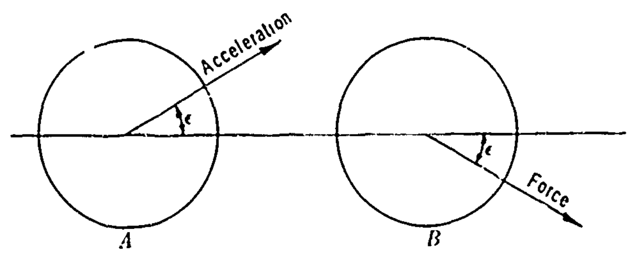


Figure 161 – Illustrating direction of force due to acceleration of another slender cylinder. See Section 95(a).

(2) Motion of either cylinder at velocity V relative to the fluid at infinity, in a direction inclined at an angle α to the line of axes drawn toward the other cylinder, evokes a force upon the other whose magnitude is $4\pi\rho a^2 b^2 V^2/r^3$. The direction of this force makes an angle 2α with the line of axes drawn as described, but it lies on the opposite side of this line from the direction of the velocity V. This force is shown as F_V in Figure 160. For, when $\theta = 0$, $\dot{x}_1^2 - \dot{y}_1^2 = V^2 \cos 2\alpha$, $2\dot{x}_1\dot{y}_1 = V^2 \sin 2\alpha$.

In particular, a slender cylinder moving toward or away from another ($\alpha = 0$ or π), or stationary but immersed in a stream that is uniform at infinity and directed parallel to the line of axes, repels the other cylinder, whereas, if the motion of the cylinder or of the stream is transverse to the line of axes, the force is attractive.

If more than two cylinders are present, the forces due to the motion of each are simply additive, in this approximation.

(B) Slender Cylinder and a Wall

If $b = a$, if the fluid is at rest at infinity, and if at all times $\beta = \pi - \alpha$ and $W = V$, then it is obvious from symmetry that the plane bisecting the line joining the axes of the two cylinders remains a stream surface. A rigid wall may be inserted along this plane, and cylinder A and the fluid on its side of the wall may be discarded.

Then the formulas may be taken to refer to a cylinder of radius a moving with its axis distant $h = r/2$ from a rigid parallel wall; a/h is to be assumed small. See Figure 162.

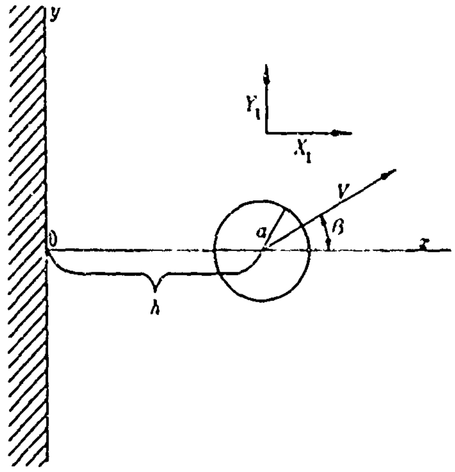


Figure 162 – Slender cylinder moving near a wall. See Section 95(B).

The cylinder has a velocity V inclined at an angle β to the positive x-axis, which is drawn perpendicularly away from the wall. The kinetic energy per unit length is half of T_1 as given by Equation [95c] or

$$T_1 = \frac{\pi}{2} \rho a^2 V^2 \left(1 + \frac{a^2}{2h^2}\right). \qquad [95g]$$

Let the components of the acceleration of the cylinder be denoted by $\ddot{x}$ and $\ddot{y}$. Then in Equation [95e, f]

$$\ddot{x}_1 = -\ddot{x}_2 = -\ddot{x},\ \ddot{y}_1 = \ddot{y}_2 = \ddot{y},\ \dot{x}_1 = -V\cos\beta,\ \dot{y}_1 = V\sin\beta;$$

and the component of force X_1 on unit length of the cylinder, taken positive away from the wall, and the component Y_1 taken parallel to it and toward positive β are, from Equations [95e, f],

$$X_1 = -\pi\rho a^2 \left(1 + \frac{a^2}{2h^2}\right)\ddot{x} + \frac{\pi\rho a^4 V^2}{2h^3} \cos 2\beta, \qquad [95h]$$

$$Y_1 = -\pi\rho a^2 \left(1 + \frac{a^2}{2h^2}\right)\ddot{y} + \frac{\pi\rho a^4 V^2}{2h^3} \sin 2\beta. \qquad [95i]$$

The wall thus appears to repel a cylinder that is moving toward or away from it ($\beta = 0$ or π), but to attract one that is moving parallel to it ($\beta = \pi/2$ or $3\pi/2$).

A flow parallel to the wall and uniform at infinity may be added. Then the velocity of the cylinder is to be taken relative to the fluid at infinity. If the cylinder is stationary, the stream pushes it toward the wall.

For the general case of two cylinders, see the references in Section 94 or Basset.[5]

96. TWO OR THREE LAMINAS

The noncirculatory flow past two plane laminas, placed with their central axes parallel and with their faces perpendicular to the plane through their axes, has been treated by Nomura.[128] The torques on the laminas tend to turn them perpendicular to the incident stream. The resultant forces on the two laminas are equal and opposite; if the laminas are of equal width, the forces tend to separate them and are proportional to $\cos^2 \phi$ where ϕ is the angle between the direction of the stream and the normal to their faces. See also Ferrari.[129]

The flow past a similar pair of laminas of equal width, with or without circulation, has been studied with reference to biplane theory by Kutta,[76] Schmitz[130] and Munk.[131] Lines of flow for two positions of the laminas, in a stream perpendicular to their faces, are shown in Figure 163.

Three parallel laminas were studied by Tani.[132]

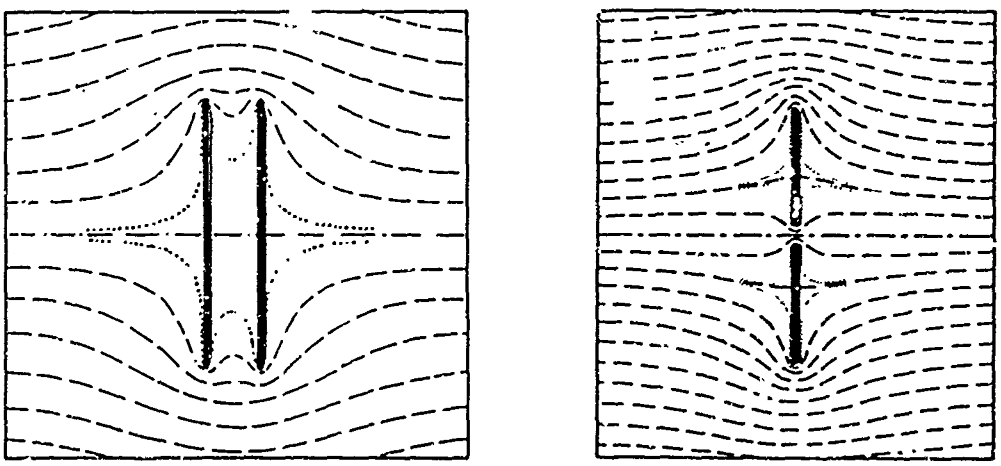


Figure 163 – Streamlines past two similar plane laminas in two positions. (Copied from Reference 133.)

97. GRATINGS

A linear grating consists of a set of parallel cylinders or laminas of any shape, usually infinite in number, all alike and similarly oriented, with their centers equally spaced along a straight line. Examples occur in Sections 46 and 47.

Another simple case that can be treated with elementary [illegible] is the following:

$$z = \frac{a}{\pi} \ln \cos \frac{\pi w}{aV} , \qquad [97a]$$

$$x = \frac{a}{2\pi} \ln \left[\frac{1}{2} \left(\cos \frac{2\pi\phi}{aV} + \cosh \frac{2\pi\psi}{aV} \right) \right] , \qquad [97b]$$

$$y = - \frac{a}{\pi} \tan^{-1} \left(\tan \frac{\pi\phi}{aV} \tanh \frac{\pi\psi}{aV} \right), \qquad [97c]$$

$$u = \frac{V}{2G} \sin \frac{2\pi\phi}{aV} , \; v = \frac{V}{2G} \sinh \frac{2\pi\psi}{aV} , \qquad [97d, e]$$

$$G = \sin^2 \frac{\pi\phi}{aV} + \sinh^2 \frac{\pi\psi}{aV} = \frac{1}{2} \left(\cosh \frac{2\pi\psi}{aV} - \cos \frac{2\pi\phi}{aV} \right) . \qquad [97f]$$

The field represented by these formulas is clearly periodic in the direction of y, which changes by a when ϕ is increased by aV, while x, u, and v return to their initial values.

Assuming such a definition of $\tan^{-1}$ that $\phi = 0$ and $\psi = 0$ at $x = y = 0$, let ψ for the moment be kept equal to zero. Then, as ϕ increases from zero, y remains zero but x decreases, down to $-\infty$ at $\phi = aV/2$; there, let the $\tan^{-1}$ in y decrease discontinuously by π; then, as ϕ increases further, z returns along the line $y = a$ to the point $(0, a)$ where $\phi = aV$, then recedes again along the same line, returns after a further decrease of $\tan^{-1}$ by π along $y = 2a$, and so on. That this interpretation of $\tan^{-1}$ is in harmony with continuity is seen by assuming ψ to be slightly less than zero; then z recedes a long way toward $x = -\infty$, $\tan^{-1}$ decreases rapidly almost to $-\pi$ as ϕ passes through the value $aV/2$, then z returns slightly below $y = a$, swings around $(0, a)$ as ϕ increases past the value aV, recedes again slightly above $y = a$, and so on. Negative ϕ corresponds similarly to negative y.

The real axis of w is thus transformed into an infinite set of lines parallel to x, spaced a apart and each extending from $x = -\infty$ to $x = 0$. A set of rigid planes may be inserted along these lines, forming a plane grating having plates of infinite width.

As ψ decreases to a large negative value,

$$x \to +\infty, \; u \to 0, \; v \to -V, \text{ since } \sinh/\cosh \to -1.$$

Thus, if ψ is assumed to be negative, the formulas represent flow past the grating with a velocity V toward negative y at distant points. A few of the streamlines are shown in Figure 164.

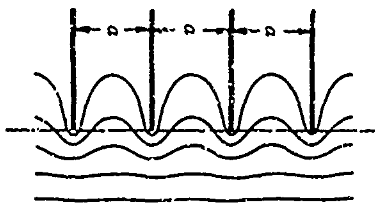


Figure 164 – Streamlines past a grating or grille consisting of semi-infinite plates symmetrically placed. (Copied from Reference 133.)

The penetration into the grating spaces is small. On one of the plates, where $\psi = 0$ and $x < 0$,

$$u = V \cot \frac{\pi\phi}{aV} = \pm V e^{\frac{\pi x}{a}} \bigg/ \left(1 - e^{\frac{2\pi x}{a}}\right)^{\frac{1}{2}}, \qquad [97g]$$

after solving Equation [97b] for ϕ in terms of x.

A uniform stream at velocity U parallel to the plates can easily be added. Then a representation is obtained of a stream that approaches at velocity $\sqrt{U^2 + V^2}$ and at an angle $\tan^{-1}(V/U)$ with the plates, and flows off between them. Part of the streamlines and flow net for such a case are shown in Figure 165, in which the plates extend off toward the right.

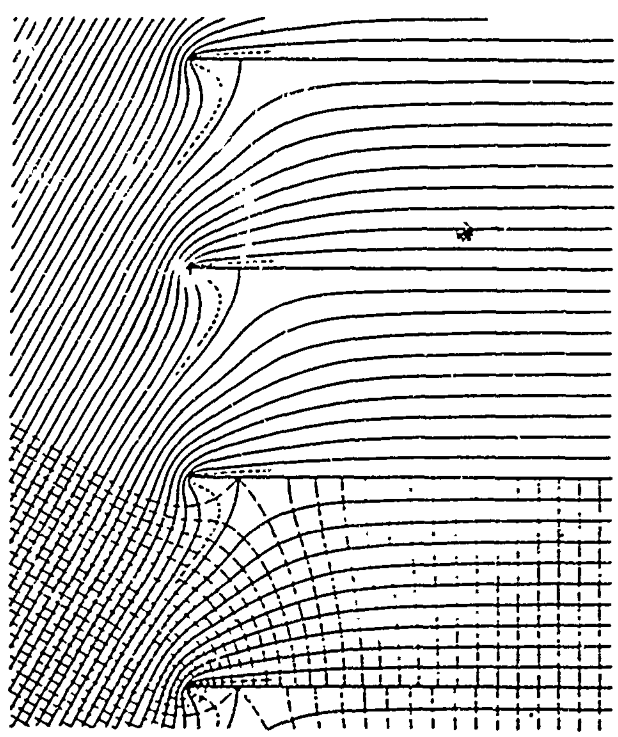


Figure 165 – Oblique flow (without circulation) past and through a grating as described under Figure 164. (Copied from Reference 135.)

The transformation Equation [97a] can be arrived at by first transforming the grating into a cylinder represented by the unit circle on the t-plane, using $z = (a/\pi) \ln [(t + 1/t)/2]$, and then assuming $w(t)$ to represent circulatory flow around this cylinder. This is done by Prandtl.[133]

A similar infinite grating with plates of finite width and flow past both sides was treated by Kutta,[76] Grammel,[134] and Engel.[135] Other cases are considered by Steuding,[136] Ringleb,[137] Konig,[138] and, with direct reference to turbine theory, by Busemann[139] and Weinel;[140] see also Sedov, Reference 141.

Circular Gratings

Gratings may also be constructed with circular symmetry, so as to have the property that the grating coincides with itself after a certain integral fraction of a turn about the axis of symmetry. The flow through such gratings has been studied as a basis for applications in the theory of turbines and centrifugal pumps. For this purpose a line source and vortex may be assumed to exist on the axis, also circulation about the exterior of the grating, and perhaps other line vortices disposed with the symmetry of the grating. References: Spannhake,[142] Schulz,[143] Florin,[144] and other references there given.

98. VORTICES NEAR CYLINDERS OR WALLS

In addition to the simple cases treated in Section 42, many other cases have been studied of a line vortex in the presence of a rigid cylinder or wall. Usually the center of interest has lain in the motion of the vortex itself, which is assumed to move with the fluid. The following may be noted:

Vortex near a lamina: Cisotti,[145] Paul,[45,46] and Caldonazzo.[146]

Vortex near a slit in an infinite plane: Paul.[45]

Vortices near a broken wall or in a channel of varied width: Mazet,[147] Müller,[41,48] Miyadzu,[149] and Zeuli.[150]

Vortex near a semicircular lamina or near a wall with a semicircular boss: De.[151]

Vortex inside a semicircle: Cisotti.[152]

Vortex inside a rectangular cylinder: Jaffé,[30] Müller,[148] and Seth.[153]

Vortex inside a curvilinear rectangle: Greenhill,[26] and Kondo.[154]

Vortex near an elliptic cylinder or inside an elliptic shell: Coates,[155] and Rosenhead;[156] also Caldonazzo,[32] Poggi,[157] Sanuki and Arakawa,[158] and Tomotika.[159]

Vortex near a parabolic cylinder: Masotti.[160]

Vortex near a cylinder of certain other shapes: Caldonazzo,[32] a cardioid,[161] where the force is questionable; Morris.[29]

Vortices between parallel walls: Jaffé[30] and Masotti;[162] trains of vortices, Glauert,[163] Rosenhead,[164] Tomotika,[165] Imai,[166] and Schwarz.[167]

Two vortices and two parallel laminas: Riabouchinsky[168] and Villat.[169]

Source and vortex near a plane lamina: Cisotti[145] and Agostinelli.[170] By using two out of the three elements source, vortex, and circulation the velocity can be made finite on both edges of the lamina.

ROTATING BOUNDARIES

99. MOVING BOUNDARIES

From the properties of the stream function ψ, the required condition at a moving boundary is readily seen to take the form

$$\frac{\partial \psi}{\partial s} = q_n. \qquad [99a]$$

Here $\partial\psi/\partial s$ is the space rate of change of ψ along the boundary in a chosen positive direction; q_n is the component of the velocity of the boundary in the direction of its normal, taken as positive when directed toward the side that lies on the left as the boundary is traced in the positive direction; compare Figure 166a. If the boundary is at rest, $q_n = 0$, hence $\partial\psi/\partial s = 0$ and, as hitherto assumed, ψ is constant.

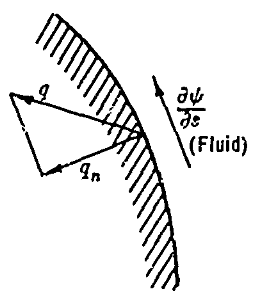


Figure 166a

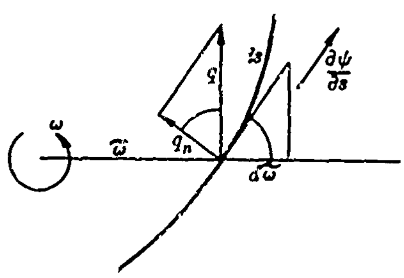


Figure 166b

Figure 166 – Relations at a moving boundary. See Section 99.

In any given field of irrotational flow, a physical boundary may be supposed inserted along any chosen curve provided the boundary is assumed to move at every point as is required by Equation [99a]; the flow will then be undisturbed by the insertion of the boundary. In this way the flow around moving boundaries of many forms can be found.

If the boundary moves in translation, an alternative procedure is the familiar one of first solving the problem with the boundary at rest and then imposing an additional uniform velocity upon everything. A more useful application of Equation [99a] is to rotating cylinders.

If a rigid cylinder of any shape rotates at angular velocity ω about an axis parallel to its generators, the velocity of its surface at any point distant ϖ from the axis has a magnitude $q = \omega\varpi$ and is directed at right angles to the radius. Hence, by similar triangles, as illustrated in Figure 166b,

$$\frac{q_n}{q} = \frac{q_n}{\omega\varpi} = \frac{d\varpi}{ds},$$

where $d\varpi$ is the increment of ϖ corresponding to ds. Using this value of q_n and Equation [99a],

$$d\psi = ds\,\frac{\partial\psi}{\partial s} = \omega\varpi\,\frac{d\varpi}{ds}\,ds = \omega\varpi d\varpi,$$

whence

$$\psi - \frac{1}{2}\,\omega\varpi^2 = C \qquad [99b]$$

after integrating. Or, if the origin is taken on the axis, so that $\varpi^2 = x^2 + y^2$,

$$\psi - \frac{1}{2}\,\omega\,(x^2 + y^2) = C = \text{constant}. \qquad [99c]$$

If any known stream function is inserted for ψ in this equation and any chosen value of C, the equation defines a certain curve. A rigid cylinder or shell may be inserted along this curve; then ψ and the associated potential ϕ will represent a possible flow around the cylinder, or inside the shell, when it is rotating at angular velocity ω about the axis from which ϖ is measured.

The velocity of the fluid relative to the cylinder or shell, or relative to axes rotating with it, may be of interest. Let q_z, q_{ϖ}, q_θ denote components of the velocity in the directions of cylindrical coordinates z, ϖ, θ, where the axis of z is drawn along the axis of rotation and the angle θ is measured around it in the positive direction of rotation; and let q_z', q_{ϖ}', q_θ' denote the corresponding components of the relative velocity. Then any point sharing in the rotation has a velocity $\omega\varpi$ in the direction of q_θ. Hence

$$q_z' = q_z, \quad q_{\varpi}' = q_{\varpi}, \quad q_\theta' = q_\theta - \omega\varpi. \qquad [99d, e, f]$$

(See Reference 1, Art. 71, 72.)

100. ROTATING CHANNEL

Consider the channel between two parallel plane walls of infinite extent which are rotating at velocity ω about an axis drawn parallel to the walls. Let there be no component of the fluid velocity in the direction of the axis. The walls will be represented on the xy-plane by two parallel lines; let these lie at distances a_1, [illegible] om the axis of rotation. Take the origin on the axis of rotation, and draw the x-axis [illegible] dicular to the walls, as in Figure 167.

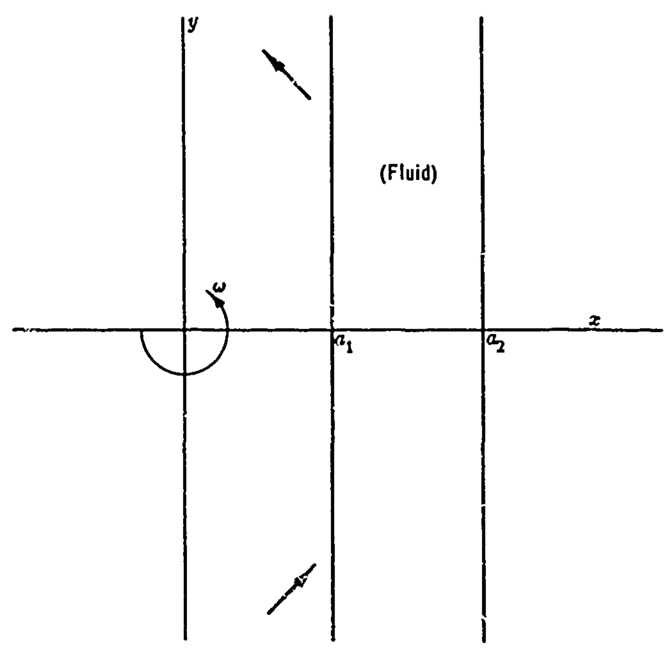


Figure 167 – A channel or infinite box in rotation. See Section 100.

Then the equation of either wall will be of the form, x = constant. It follows that in Equation [99c] y must cancel out. This condition is met if

$$\psi = -\frac{\omega}{2}(x^2 - y^2) + Ax \qquad [100a]$$

where A is an arbitrary constant. This is a permissible form for ψ, since the last term represents uniform motion at velocity A toward positive y, and the first term is adapted from Equation [36e]. Using also Equation [36d], the corresponding potential and components of velocity are

$$\phi = \omega xy - Ay, \quad u = -\omega y, \quad v = -\omega x + A. \qquad [100b, c, d]$$

Since the velocity of a point rotating with the walls has components

$$u = -\,\omega y, \quad v = \omega x,$$

the velocity of the fluid relative to the walls has components, taken in the instantaneous directions of the rotating axes,

$$u' = 0, \quad v' = -\,2\,\omega\, x + A.$$

Thus in the relative motion the lines of flow are straight and parallel to the walls.

The constant A is connected with the total flow through the channel; the volume passing per second relatively to the walls, per unit of length perpendicular to the planes of flow, is

$$Q' = \int_{a_1}^{a_2} v'dx = (a_2 - a_1)\,[A - \omega\,(a_1 + a_2)].$$

The relative velocity varies linearly from one wall to the other; it may vanish on an intermediate plane and have opposite directions near the two walls; see Reference 10, page 79.

101. ROTATING ANGLE

Consider the irrotational two-dimensional motion of the fluid in an angular space formed by two semi-infinite plane walls meeting at a fixed angle 2α. Let the walls rotate at constant velocity ω about their line of intersection. With the origin taken on this line and the x-axis drawn along the bisector of the angle, the walls will be represented on the xy-plane by two radii drawn from the origin at $\theta = \pm\alpha$, where $\theta = \tan^{-1}(y/x)$.

The following assumption, suggested by the conjugate flow of Section 36, will be found to satisfy Equation [99b] on the walls, with $C = 0$, since here $\tilde{\omega} = r$:

$$\phi = -\frac{1}{2}\,\omega r^2\,\frac{\sin 2\theta}{\cos 2\alpha}, \quad \psi = \frac{1}{2}\,\omega r^2\,\frac{\cos 2\theta}{\cos 2\alpha}. \qquad [101a, b]$$

Then

$$q_r = \omega r\,\frac{\sin 2\theta}{\cos 2\alpha}, \quad q_\theta = \omega r\,\frac{\cos 2\theta}{\cos 2\alpha}, \quad q = \frac{|\omega|\, r}{\cos 2\alpha}. \qquad [101c, d, e]$$

For the velocity relative to the walls, by Equations [99e, f],

$$q_r' = \omega r\,\frac{\sin 2\theta}{\cos 2\alpha}, \quad q_\theta' = \omega r\left(\frac{\cos 2\theta}{\cos 2\alpha} - 1\right). \qquad [101f, g]$$

If, as will be assumed, $\alpha < \pi/4$, q_r' has the same sign as has the product $\omega\theta$; the fluid thus flows inward or toward the axis along the trailing wall of the angle and outward along the leading wall. This is illustrated by Figure 168, in which relative streamlines are shown for $\alpha = 9$ deg. If $\alpha > \pi/4$, the flow pattern is more complicated, but such cases are of little practical interest. If $\alpha = \pi/4$, $\cos 2\alpha = 0$ and irrotational motion is impossible, in the ideal case in which the walls extend to infinity.

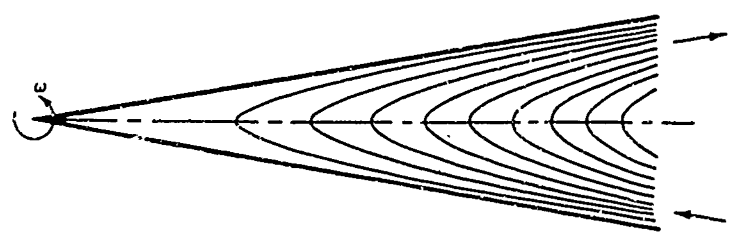


Figure 168 – Streamlines for the motion of fluid relative to the walls in a rotating angle. (Copied from Reference 10.)

The pressure in the fluid, from Equation [11d], is

$$p = \rho\, \omega^2 r^2 \left(\frac{\cos 2\,\theta}{\cos 2\alpha} - \frac{1}{2\cos^2 2\alpha}\right) + p_0. \qquad [101h]$$

The flow within the angle can be generalized by adding one or more terms of the following form, derivable from a complex potential

$$w = i\,\omega\, z^{(2n+1)\,\pi/2\alpha}:$$

$$\phi_n = -A_n\, r^{(2n+1)\frac{\pi}{2\alpha}} \sin\left[(2n+1)\frac{\pi\theta}{2\alpha}\right],$$

$$\psi = A_n\, r^{(2n+1)\frac{\pi}{2\alpha}} \cos\left[(2n+1)\frac{\pi\theta}{2\alpha}\right], \qquad [101i,j]$$

where n is any positive integer and A_n is an arbitrary real constant. The corresponding contribution to q_θ vanishes at $\theta = \pm\alpha$, so that the boundary condition, $q_\theta = \omega r$, remains satisfied.

Rotation Combined with Outflow

With reference to applications in turbine theory, it is of interest to imagine a line source to exist along the apex of the rotating angle. By the principal of superposition, the source adds a radial component of velocity A/r where A is a constant, so that, in place of Equation [101c] or Equation [101f],

$$q_r = q_r' = \omega r \frac{\sin 2\theta}{\cos 2\alpha} + \frac{A}{r}; \qquad [101k]$$

q_θ and q_θ' are unchanged. A volume $2\alpha\ A$ of fluid flows outward through the angle, per second and per unit of length perpendicular to the planes of motion.

A stagnation line now occurs on one face of the angle, on the rear face if ω and A have the same sign, at $r = r_0 = (A/\omega \tan 2\alpha)^{1/2}$. Where $r < r_0$, q_r' has everywhere the same sign, but beyond this point reversal of the radial velocity occurs from one side of the angle to the other, as it does in the absence of outflow. As r increases, the outflowing fluid that has come from the source becomes crowded more and more against the leading wall of the angle. If $A < 0$, there is a line sink on the axis, and the fluid that is destined to be absorbed by it is crowded against the trailing edge. The streamlines are illustrated in Figure 169.

(For notation and method; see Section 34; Reference 10, page 94)

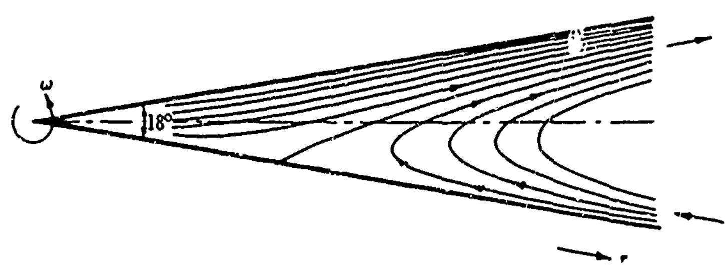


Figure 169 – Streamlines for the motion of fluid relative to the walls in a rotating angle containing a line source at its apex, i.e., on the axis of rotation. (Copied from Reference 10.)

102. FLUID WITHIN A ROTATING SECTOR

Consider the fluid within a vessel whose section has the form of a circular sector of radius a and aperture 2α; let it rotate at angular velocity ω about its edge, or about the apex of the sector; see Figure 170.

This problem differs from the last in the additional boundary condition that $q_r = 0$ at $r = a$, where r denotes distance from the axis. The solution can be constructed by adding to

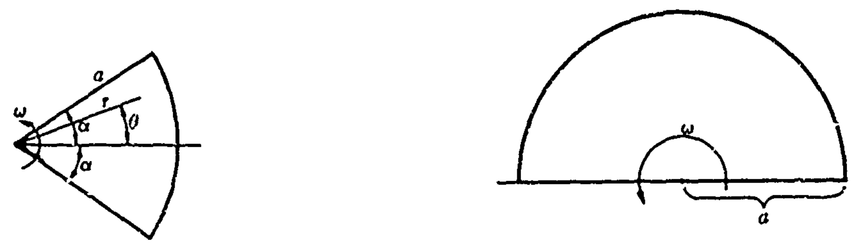


Figure 170a   Figure 170b

Figure 170 – Two sector-shaped cylindrical profiles. See Section 102.

Equation [101a, b] a Fourier series composed of terms of the form of Equation [101i, j] and giving suitable values to the coefficient A_n.

Assume that

$$\phi = -\frac{1}{2}\,\omega r^2 \frac{\sin 2\theta}{\cos 2\alpha} - \omega a^2 \sum_{n=0}^{\infty} A_{2n+1}\left(\frac{r}{a}\right)^{(2n+1)\frac{\pi}{2\alpha}} \sin\left[(2n+1)\frac{\pi\theta}{2\alpha}\right], \quad [102a]$$

where θ is measured as before from the bisector of the apical angle. Then, to make $q_r = (-\,\partial\phi/\partial r) = 0$ at $r = a$,

$$0 = \omega a \frac{\sin 2\theta}{\cos 2\alpha} + \omega a \sum_{n=0}^{\infty} (2n+1)\frac{\pi}{2\alpha} A_{2n+1} \sin\left[(2n+1)\frac{\pi\theta}{2\alpha}\right].$$

To find A_{2k+1}, multiply through by $\sin[(2k+1)\,\pi\theta/2\alpha]$ and integrate from $\theta = -\alpha$ to $\theta = \alpha$. It is found that, replacing k by n,

$$A_{2n+1} = (-1)^{n+1} \frac{32\,\alpha^2}{\pi(2n+1)[(2n+1)^2\pi^2 - 16\,\alpha^2]}. \quad [102b]$$

The complete Fourier series would include also terms containing $\cos[(2n+1)\pi\theta/2\alpha]$, but if these are included their coefficients are at once seen to be zero because of symmetry.

The corresponding expression for ψ is

$$\psi = \frac{1}{2}\,\omega r^2 \frac{\cos 2\theta}{\cos 2\alpha} + \omega a^2 \sum_{n=1}^{\infty} A_{2n+1}\left(\frac{r}{a}\right)^{(2n+1)\frac{\pi}{2\alpha}} \cos\left[(2n+1)\frac{\pi\theta}{2\alpha}\right]. \quad [102c]$$

The relative motion in a group of three such sector-shaped cylinders rotating about their common edge is shown in Figure 171.

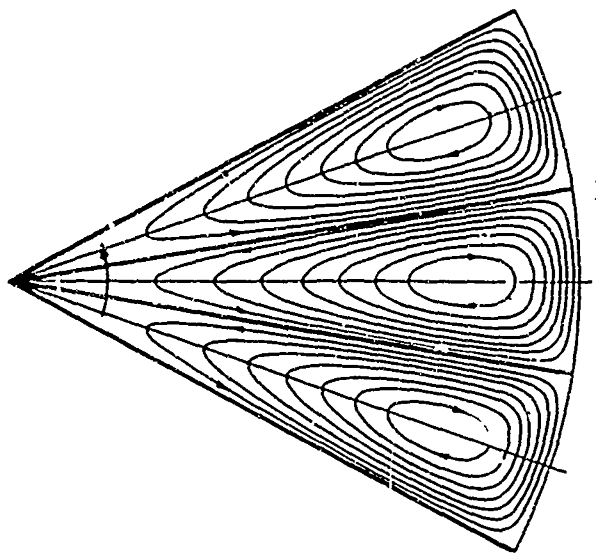


Figure 171 – Streamlines for the motion of fluid relative to the walls within a threefold sector-form cylinder rotating about its apex. See Section 102. (Copied from Reference 10.)

If $\alpha = \pi/4$ or $3\pi/4$, $\cos 2\alpha = 0$; but then, also, either the first or second A_{2n+1} becomes infinite. To handle such cases, a modified formula must be developed by seeking the limit forms that ϕ and ψ take on as α approaches the value stated.

In Equation [17d] for the kinetic energy, or $T_1 = (1/2)\rho \int \phi \, q_n \, ds$, along the side at $\theta = -\alpha$, $q_n = \omega r$, $ds = dr$, and, since $\sin [(2n+1)\pi/2] = (-1)^n$,

$$\phi = \phi_{-\alpha} = \frac{1}{2} \omega r^2 \tan 2\alpha + \omega a^2 \Sigma (-1)^n A_{2n+1} \left(\frac{r}{a}\right)^{(2n+1)\frac{\pi}{2\alpha}} .$$

On the other side, where $\theta = +\alpha$, ϕ and q_n are both reversed in sign, hence the integral has the same value as on the first side. Finally, over the curved end $q_n = 0$. Hence the kinetic energy per unit length is $T_1 = \rho\,\omega \int_0^a \phi_{-\alpha} r \, dr$, or, after integrating,

$$T_1 = \rho\,\omega^2 a^4 \left\{ \frac{1}{8} \tan 2\alpha + \sum_0^\infty (-1)^n \left[(2n+1) \frac{\pi}{2\alpha} + 2 \right]^{-1} A_{2n+1} \right\} \qquad [102d]$$

A *semicircle* is obtained if $\alpha = \pi/2$; it is revolving about the central line of its base, as in Figure 170b. In this case, since $(-1)^{2n} = 1$, using Equation [102b],

$$T_1 = \frac{1}{\pi} \rho \omega^2 a^4 \sum_0^\infty \frac{8}{(2n+3)(2n+1)[4-(2n+1)^2]}$$

$$= \frac{\rho \omega^2 a^4}{4\pi} \sum_0^\infty \left(-\frac{1}{2n-1} + \frac{4}{2n+1} - \frac{3}{2n+3} - \frac{4}{(2n+3)^2} \right).$$

Of the four series into which the last sum may be broken up, the terms of the first three cancel each other except for one term out of the second, which equals 4. Furthermore

$$\sum_0^\infty \frac{1}{(2n+3)^2} = -1 + \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \ldots\ldots = -1 + \frac{\pi^2}{8};$$

see Carslaw's Fourier Series,[17] 3d ed. (1930), p. 235. Hence

$$T_1 = \frac{\pi}{4} \rho a^4 \omega^2 \left(\frac{8}{\pi^2} - \frac{1}{2} \right) = 0.1553\, a^2 \left(\frac{1}{2} \pi \rho a^2 \right) \omega^2. \qquad [102e]$$

(See Reference 1, Article 72; Reference 10, page 102.)

103. MOTION WITHIN A ROTATING TRIANGULAR PRISM.

If

$$w = \phi + i\psi = iAz^3, \quad z = re^{i\theta}, \qquad [103a, b]$$

then

$$\phi = -Ar^3 \sin 3\theta, \quad \psi = Ar^3 \cos 3\theta, \qquad [103c, d]$$

or

$$\psi = A(x^3 - 3xy^2)$$

from

$$z^3 = (x+iy)^3, \quad \text{where} \quad x = r\cos\theta, \quad y = r\sin\theta.$$

Substituting for ψ in Equation [99c],

$$A(x^3 - 3xy^2) - \frac{1}{2}\omega(x^2+y^2) = C.$$

This is satisfied for all values of y if $x = a$ and

$$3Aa + \frac{1}{2}\omega = 0.$$

Hence, if these equations hold, the line $x = a$ may form part of a rotating boundary. Since it is evident from Equation [103c,d] that everything repeats when θ is increased by 120 deg or 240 deg, two other similar lines must exist. These three lines will enclose an equilateral triangle centered at the origin.

Substituting for A in Equation [103c,d],

$$\phi = \frac{\omega}{6a} r^3 \sin 3\theta, \quad \psi = -\frac{\omega}{6a} r^3 \cos 3\theta; \qquad [103e,f]$$

whence

$$q_r = -\frac{\omega}{2a} r^2 \sin 3\theta, \quad q_\theta = -\frac{\omega}{2a} r^2 \cos 3\theta, \quad q = \frac{|\omega|}{2a} r^2. \qquad [103g,h,i]$$

These formulas represent the flow inside a vessel in the form of an equilateral triangular prism, rotating with angular velocity about an axis parallel to its length and passing through the center of its section. The vertices are at $r = 2a$ and $\theta = 60$ deg, 180 deg, and 300 deg; the sides are of length $s = 2(2a \cos 30^0)$ or $s = 2\sqrt{3}a$. The instantaneous streamlines are illustrated in Figure 172.

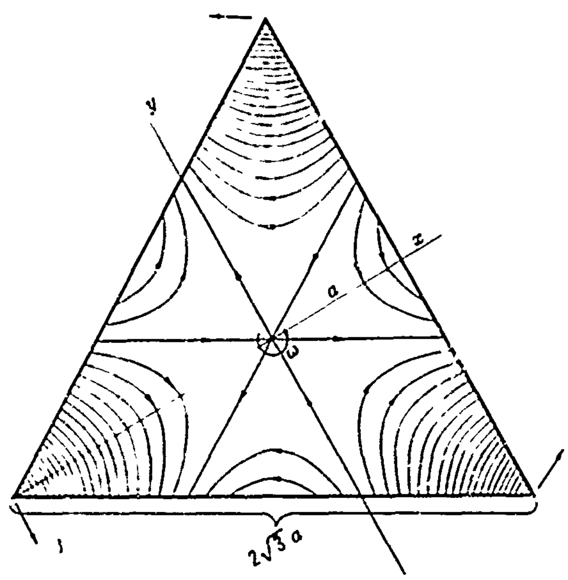


Figure 172 – Absolute streamlines for fluid within a triangular prism rotating about its axis.

The pressure p, when ω is constant, is given by Equation [11c] with $\varpi = r$ or, using Equations [103h,i],

$$p = -\frac{\rho\omega^2}{8a^2}(r^4 + 4ar^3 \cos 3\theta + p_0). \qquad [103j]$$

The kinetic energy of the fluid per unit length is

$$T_1 = \frac{1}{2}\rho \iint q^2\, dx\, dy = \frac{1}{80\sqrt{3}} \rho s^4 \omega^2. \qquad [103k]$$

(For notation and method: see Section 34; Reference 1, Article 72; Reference 2, Section 9.72.)

104. TWO COAXIAL CYLINDERS.

In the flow described at the end of Section 67, at any given value of r such as $r = b$ the radial velocity varies with θ in the same way as does the normal component of the velocity of a moving circular cylinder. Let such a cylinder, of radius b, be inserted with its axis at the origin, and let it move perpendicularly to its axis with a velocity V toward $\theta = 0$ where

$$V = U\left(-1 + \frac{a^2}{b^2}\right). \qquad [104a]$$

Then the surface of the cylinder and the fluid have the same radial component of velocity, so that the necessary boundary condition is satisfied at the surface of the cylinder.

Assume that $a > b$. Then the formulas of Section 67 will represent the flow between two coaxial cylinders of which the outer, of radius a, is stationary. After substituting from Equation [104a] for U in terms of V, Equations [67b, c, i, j] give for the motion of the fluid between the cylinders

$$\phi = \frac{b^2 V}{a^2 - b^2}\left(r + \frac{a^2}{r}\right)\cos\theta, \quad \psi = \frac{b^2 V}{a^2 - b^2}\left(r - \frac{a^2}{r}\right)\sin\theta, \qquad [104b, c]$$

$$q_r = \frac{b^2 V}{a^2 - b^2}\left(\frac{a^2}{r^2} - 1\right)\cos\theta, \quad q_\theta = \frac{b^2 V}{a^2 - b^2}\left(\frac{a^2}{r^2} + 1\right)\sin\theta. \qquad [104d, e]$$

In Equation [17d] for the kinetic energy, the contribution of the outer boundary at $r = a$ vanishes, since $q_n = 0$. Hence the kinetic energy of the fluid per unit length of the cylinders, at the instant at which they are coaxial, is

$$T_1 = \frac{1}{2}\rho\int \phi\, q_n\, ds = \frac{1}{2}\rho\int_0^{2\pi} \phi\, q_r\, b\, d\theta = \frac{1}{2}\pi\rho b^2 \frac{a^2 + b^2}{a^2 - b^2} V^2, \qquad [104f]$$

after substituting the values of ϕ and q_r with $r = b$ and integrating.

105. FLUID WITHIN A ROTATING SHELL OF ELLIPTIC OR OTHER SHAPE

If Equation [99c] in Section 99 is to represent an ellipse, ψ must be a quadratic function of x and y. Consider, therefore,

$$w = \phi + i\psi = iAz^2 = iA(x + iy)^2, \quad \phi = -2Axy, \quad \psi = A(x^2 - y^2).$$

Substituting for ψ in Equation [99c],

$$A(x^2 - y^2) - \frac{\omega}{2}(x^2 + y^2) = C.$$

This can be written

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a^2 = -\frac{2C}{\omega - 2A}, \quad b^2 = -\frac{2C}{\omega + 2A}. \qquad [105a, b, c]$$

These equations represent a real ellipse with semiaxes a and b provided $C < 0$, $|A| < \omega/2$. Substituting for A in terms of ω, a^2, and b^2,

$$\phi = -\omega k\, xy, \; \psi = \frac{1}{2}\omega k(x^2 - y^2), \quad k = \frac{a^2 - b^2}{a^2 + b^2}. \qquad [105d, e, f]$$

These formulas represent the flow inside a cylindrical shell whose cross section is an ellipse of semiaxes a and b, when it is rotating at angular velocity ω about its axis, on which the origin of coordinates has been taken. The coordinate axes must be allowed to rotate with the shell, but all velocities are referred to a fixed frame of reference. The streamlines are rectangular hyperbolas. They are illustrated in the interior of the heavy ellipse drawn in Figure 173, which may be taken to represent half of the symmetrical shell.

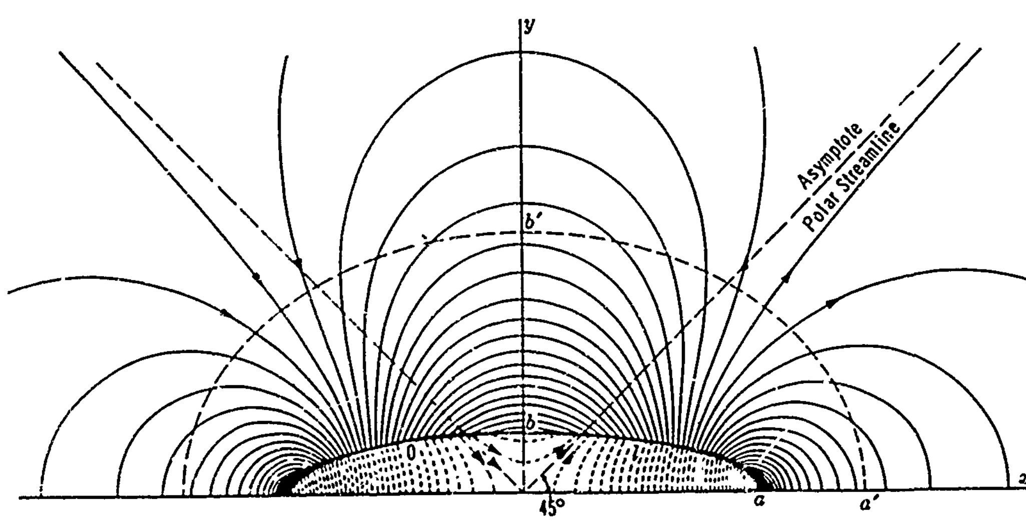


Figure 173 – The heavy ellipse may represent half of the contour of an elliptic cylinder that is rotating about its axis and producing streamlines represented by the solid curves, or half of an elliptic shell rotating similarly and producing streamlines in the contained fluid represented by the broken curves. See Section 106.
(Copied from Reference 174.)

In terms of cylindrical coordinates ϖ, θ such that $x = \varpi \cos \theta$, $y = \varpi \sin \theta$,

$$\phi = -\frac{1}{2}\, \omega k \varpi^2 \sin 2\theta, \quad \psi = \frac{1}{2}\, \omega k \varpi^2 \cos 2\theta; \qquad [105g, h]$$

hence

$$q_{\varpi} = \omega k \varpi \sin 2\theta, \quad q_\theta = \omega k \varpi \cos 2\theta, \qquad [105i, j]$$

$$q = |\omega|\, k \varpi. \qquad [105k]$$

At the end of the minor axis, $\theta = \pi/2$, $q_{\varpi} = 0$, $q_\theta = -\omega k \varpi$, so that the fluid is circulating backwards. At the end of the major axis, $\theta = 0$, $q_\theta = \omega k \varpi$; but the velocity of the shell is $\omega \varpi$ and so exceeds q_θ, since $k < 1$. Thus relatively to the shell the fluid circulates in the opposite direction, in order to keep its motion irrotational in space.

The pressure at any point, if the rotation is steady, is, from Equation [11c],

$$p = \rho \omega^2 k \varpi^2 \left(\cos 2\theta - \frac{1}{2}\, k\right) + \text{constant}. \qquad [105l]$$

The kinetic energy of the fluid, per unit length of the cylinder, is

$$T_1 = \frac{1}{2}\, \rho \iint q^2\, dx\, dy,$$

where the boundary of the region of integration is the ellipse defined by Equation [105a]. Substituting $x = a x'$, $y = b y'$, and the value of q,

$$T_1 = \frac{1}{2}\, \rho\, ab\, \omega^2 k^2 \iint (a^2 x'^2 + b^2 y'^2)\, dx'\, dy',$$

and the region of integration for x' and y' is now a circle of unit radius. Changing to polar coordinates so that $x' = r \cos \theta$, and replacing $dx'\, dy'$ by $r\, dr\, d\theta$,

$$\iint y'^2\, dx'\, dy' = \iint x'^2\, dx'\, dy' = \int_0^1 r^3\, dr \int_0^{2\pi} \cos^2 \theta\, d\theta = \frac{\pi}{4}\,.$$

Hence

$$T_1 = \frac{\pi}{8}\, \rho k^2\, ab\, (a^2 + b^2)\, \omega^2. \qquad [105m]$$

If the fluid rotated as a rigid body, T_1 would be $\pi\rho ab(a^2 + b^2)\omega^2/8$.

For the elliptic shell; see Reference 1, Articles 71, 72. A shell whose cross section is an elliptic quadrant was studied by Sen,[171] one composed of arcs and lines, by Ghose.[172]

(For notation and method; see Section 34.)

106. ROTATION OF ELLIPTIC CYLINDER OR LAMINA.

A transformation that yields a quadratic stream function with the fluid at rest at infinity is that of Section 84 with $\Gamma = 0$ or, after a slight change of notation,

$$w = i\,A\,e^{-2\zeta}, \quad z = c\cosh\zeta, \qquad [106a, b]$$

where $z = x + iy$, $\zeta = \xi + i\eta$ and the elliptic coordinates ξ and η on the z-plane are described in Section 82. Here $\xi \geqq 0$ and η is multiple valued like an angle. The ellipse for a given ξ has semiaxes a', b' such that

$$a' = c\cosh\xi, \quad b' = c\sinh\xi, \quad a' + b' = c\,e^{\xi}, \quad c = \sqrt{a'^2 - b'^2}\,. \qquad [106c, d, e, f]$$

$$x = c\cosh\xi\cos\eta = a'\cos\eta, \quad y = c\sinh\xi\sin\eta = b'\sin\eta. \qquad [106g, h]$$

From $w = \phi + i\psi$

$$\phi = A\,e^{-2\xi}\sin 2\eta, \quad \psi = A\,e^{-2\xi}\cos 2\eta. \qquad [106i, j]$$

From Equation [106g, h] and a hyperbolic formula in Section 32, $x^2 + y^2 = (c^2/2)(\cosh 2\xi + \cos 2\eta)$; substitution for $x^2 + y^2$ in Equation [99c] gives

$$A\,e^{-2\xi}\cos 2\eta - \frac{c^2}{4}\,\omega\,(\cosh 2\xi + \cos 2\eta) = C. \qquad [106k]$$

This equation is satisfied for any value of η provided $\xi = \xi_0$ = constant and

$$C = -\frac{c^2}{4}\,\omega\cosh 2\xi_0, \quad A\,e^{-2\xi_0} = \frac{c^2}{4}\,\omega.$$

Thus for this particular value of C the curve defined by Equation [106k] degenerates into the ellipse defined by $\xi = \xi_0$. Its semiaxes a, b are such that, by Equation [106e,f],

$$a + b = c\,e^{\xi_0}, \; c = \sqrt{a^2 - b^2}. \qquad [106l, m]$$

Eliminating A and ξ_0,

$$w = \frac{i}{4}\,\omega\,(a+b)^2\,e^{-2\zeta}, \qquad [106n]$$

$$\phi = \frac{\omega}{4}\,(a+b)^2\,e^{-2\xi}\sin 2\,\eta,\quad \psi = \frac{\omega}{4}\,(a+b)^2\,e^{-2\xi}\cos 2\,\eta. \qquad [106o,p]$$

From Equations [82s, t, u] the components of velocity, in directions given by Equations [82o, p], are

$$q_\xi = \frac{\omega}{2}\,\frac{(a+b)^2}{cG}\,e^{-2\xi}\sin 2\,\eta,\quad q_\eta = -\frac{\omega}{2}\,\frac{(a+b)^2}{cG}\,e^{-2\xi}\cos 2\,\eta, \qquad [106q,r]$$

$$G = (\sinh^2\xi + \sin^2\eta)^{1/2} = \left[\frac{1}{2}\,(\cosh 2\,\xi - \cos 2\,\eta)\right]^{1/2}. \qquad [106s]$$

At large distances from the origin, ξ is large and $\cosh\xi = \sinh\xi = e^{\xi}/2$, nearly, so that $x^2 + y^2 = c^2\,e^{2\xi}/4$. Thus q vanishes in proportion to $(x^2+y^2)^{-3/2}$.

On the x-axis, $\cos\eta = \pm 1$, $x = \pm c\cosh\xi$ and $u = \pm q_\xi = 0$, while v has the opposite sign to that of x and is, since $e^{-2\xi} = (\cosh\xi + \sinh\xi)^{-2}$,

$$v = \pm q_\eta = \mp\frac{\omega}{2}(a-b)\,(a+b)^3\,[\sqrt{x^2-c^2}\,(|x| + \sqrt{x^2-c^2})^2]^{-1} \qquad [106t]$$

On the y-axis, $\sin\eta = \pm 1$, $y = \pm c\sinh\xi$, and $v = \pm q_\xi = 0$, while u has a sign opposite to that of y and is

$$u = \mp q_\eta = \mp\frac{\omega}{2}\,(a-b)\,(a+b)^3\,[\sqrt{y^2+c^2}\,(|y| + \sqrt{y^2+c^2})^2]^{-1}. \qquad [106u]$$

The formulas represent the flow around an elliptical cylinder of cross-sectional semiaxes a and b, rotating about its axis at angular velocity ω, in fluid that is at rest at infinity. The origin lies on the axis of rotation, and the axes rotate with the cylinder. The velocities as given refer, however, to fixed axes; it may be supposed that the axes are momentarily stationary in their instantaneous positions.

The streamlines for $\psi = 0$ correspond to $\eta = \pm 45°$ or $\pm 135°$ and are easily seen to be asymptotic to the radii $y = \pm x$. These streamlines separate those that cross the x axis at their outer extremities from those that cross the y-axis.

On the cylinder $\xi = \xi_0$ and, from Equations [106l, m, o, p],

$$\phi = \frac{\omega}{4}(a^2 - b^2)\sin 2\eta, \ \psi = \frac{\omega}{4}(a^2 - b^2)\cos 2\eta. \qquad [106v, w]$$

The flow net is the same for all confocal cylinders; the values of ϕ and ψ at corresponding points are proportional to $(a + b)^2$. For, the values of ξ and η at a given point in space depend only on the locations of the foci, and in particular upon c.

If $b \to 0$, the ellipsoid becomes a *lamina* of width $2c = 2a$, rotating about its median axis. On its surface $\xi = 0$, $y = 0$, $x = a \cos \eta$ and

$$u = \mp q_\eta = \pm \frac{\omega}{2} \frac{2x^2 - a^2}{\sqrt{a^2 - x^2}}, \ v = \pm q_\xi = \omega x. \qquad [106x, y]$$

Here the upper sign refers to the face toward positive y, the lower sign to the other face, and ω is positive as usual for rotation from x toward y. Thus $u = 0$ at $x = \pm a/\sqrt{2}$.

Streamlines for equally spaced ψ are shown around the elliptical cylinder ab in Figure 173. The curves inside the ellipse are to be disregarded in this connection. The fluid is at rest relative to the cylinder at points o and i. The curve $a'b'$ represents a confocal cylinder that would give rise to the same streamlines at external points. The flow net for a lamina is illustrated in Figure 174. The streamlines differ in appearance in the two cases only because different spacings of ψ were chosen.

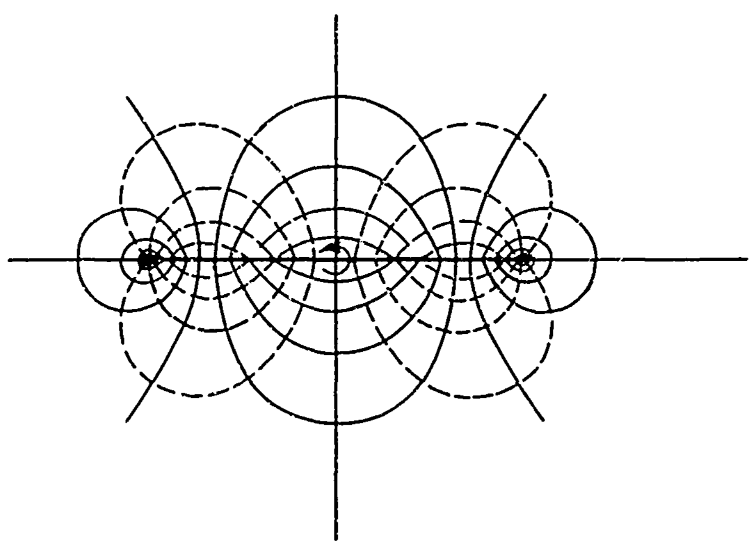


Figure 174 – Flow net produced by a plane lamina rotating about its median line. (Copied from Reference 10.)

The kinetic energy of the fluid per unit length of the cylinder is, from Equations [106v, w] and Equation [17d],

$$T_1 = \frac{\rho\omega^2}{16}(a^2 - b^2)^2 \int_0^{2\pi} \sin^2 2\eta d\eta = \frac{\pi\rho}{16}(a^2 - b^2)^2 \omega^2. \qquad [106z]$$

For a lamina ($b = 0$)

$$T_1 = \frac{1}{16}\pi\rho\,\omega^2 a^4. \qquad [106a']$$

Circulation Γ about the cylinder or lamina can be included by adding the same terms containing Γ as in Section 84.

Combined Translation and Rotation, or Rotation about Other Axes.

Any two-dimensional motion of the cylinder or lamina perpendicular to its generators can be resolved into a motion of translation and a rotation about its axis. Such a motion can also be regarded as a pure motion of rotation about some other axis. The corresponding expressions for ϕ and ψ can be constructed by adding those for the component motions, with or without circulation; and the velocity can be found by adding the two velocities vectorially.

In the case $\Gamma = 0$, when the kinetic energy is calculated by substituting the combined ϕ and ψ, as obtained from Equations [106v, w] and from Equations [84b, c] with $\xi = \xi_0$, in Equation [17d] or in $T_1 = -(\rho/2)\int\phi d\psi$, the same terms in U^2 and ω^2 are obtained as before, and in addition a product term in ωU. The latter contains, besides constants, the integral

$$\int_0^{2\pi}\{-2(b\cos\alpha\cos\eta + a\sin\alpha\sin\eta)\sin 2\eta + \cos 2\eta(-b\cos\alpha\sin\eta + a\sin\alpha\cos\eta)\}d\eta.$$

This integral, however, equals zero. Hence the kinetic energies of translation and of rotation are simply additive; their sum is the total energy.

As an example, if a lamina of width $2a$ is rotating at angular velocity ω about an axis lying in its plane, parallel to its edges but displaced a distance βa from its median line, the translation to be added is one perpendicular to its plane at velocity $U = \beta a\omega$. Hence, from Equation [84l] with $\alpha = \pi/2$ and Equation [106z], the total kinetic energy of the fluid per unit length of the lamina is

$$T_1 = \frac{1 + 8\beta^2}{16}\pi\rho\, a^4\omega^2. \qquad [106b']$$

Streamlines for this case are shown in Figure 175.

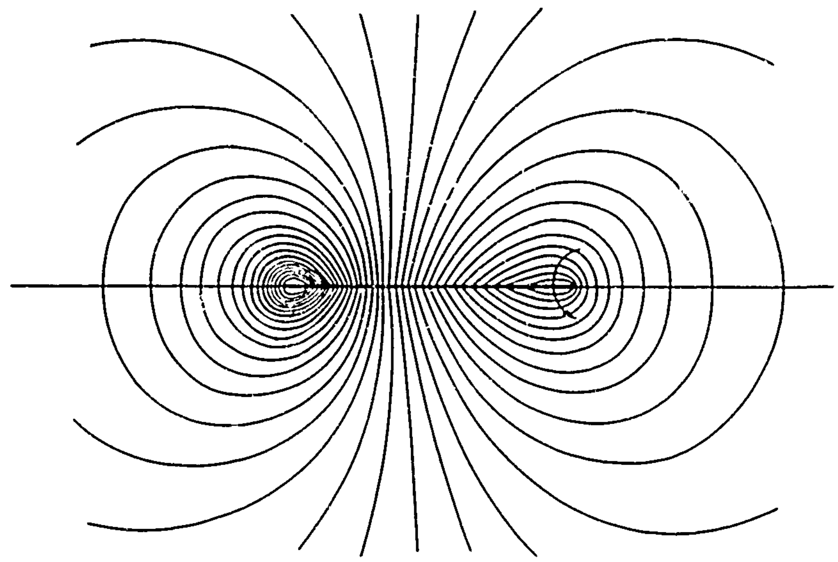


Figure 175 – Streamlines produced by a plane lamina rotating about one edge. (Copied from Reference 10.)

The pressure can be found from Equation [11c] or [11d].

A lamina bent sharply along its median line was studied by Sona.[173]

(For notation and method; see Section 34; Reference 1, Article 72; Reference 2, Section 9.65; Zahm[174] and Consiglio.[175]

CHANNELS

107. FLOW PAST A SQUARE END OR AN OFFSET

Let a stream of fluid, having a uniform velocity U at infinity, flow past an obstacle in the form of a two-dimensional semi-infinite rectangular box; let the box have a face of width $2h$ perpendicular to U and two other faces extending to infinity in the direction of U, as illustrated in cross section in Figure 176, on the z-plane.

Let the x-axis be taken in the median plane of the obstacle with the origin on its front face. Then it is obvious from symmetry that a streamline follows the x-axis to the box, divides, and proceeds to infinity along both sides of the box.

It suffices to determine the flow above the x-axis; that below it is then the mirror image in the axis of the flow above. A rigid boundary could, in fact, be inserted along the x-axis to the right of the box without disturbing the flow. Thus either half of the flow will serve also to represent the flow past a plane wall with an offset in it of width h.

The bounding streamline $COBA$ constitutes an infinite polygon and can be transformed into the real axis of a new variable t by the Schwarz-Christoffel method. The streamline must be traced in the direction $ABOC$ to make the area above it correspond to that above the

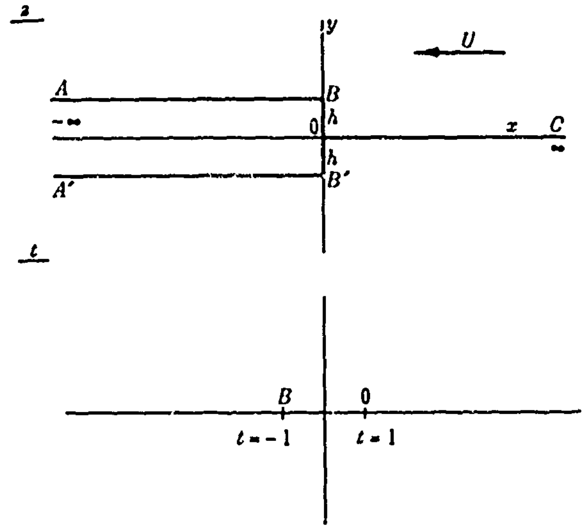


Figure 176 – A semi-infinite obstacle $AB\,B'A'$, or an offset in an infinite wall $ABOC$.

real t axis. Then the exterior angle at B is $-\pi/2$ and at 0 is $+\pi/2$. The values of t at these two points can be chosen arbitrarily as -1 and $+1$. Then, in Equation [31a] of Section 31, $a_1 = -1, \alpha_1 = -1/2$; $a_2 = 1, \alpha_2 = 1/2$; hence Equation [31a] becomes in the present case

$$\frac{dz}{dt} = K\,(t+1)^{1/2}\,(t-1)^{-1/2}. \qquad [107a]$$

Integrating,

$$z = K\{(t^2-1)^{1/2} + \ln\,[t + (t^2-1)^{1/2}]\} + L.$$

To fix the amplitudes, assume that, for the values of t required, $0 \leqq \text{amp}\ t \leqq \pi$. Then amp $(t+1)$ and amp $(t-1)$ can be understood to lie in the same range, so that amp (t^2-1) will range from 0 to 2π, and amp $(t^2-1)^{1/2}$ from 0 to π. Then, to preserve continuity, for negative real $t < -1$, $(t^2-1)^{1/2} = -\sqrt{t^2-1} < 0$. Also, $0 \leqq \text{amp}\,[t + (t^2-1)^{1/2}] \leqq \pi$, for use in defining the logarithm.

The constants K and L are chosen so as to make $t = -1$ at B or $z = ih$, and $t = 1$ at $z = 0$:

$$ih = K\log(-1) + L = i\pi K + L;\ 0 = 0 + L.$$

Hence $L = 0$, $K = h/\pi$ and

$$z = \frac{h}{\pi}\{(t^2-1)^{1/2} + \ln\,[t + (t^2-1)^{1/2}]\}. \qquad [107b]$$

The flow on the z-plane is thus transformed into a flow on the t-plane in which the real axis is a streamline. A possible complex potential for such flow is $w = ct$. With this potential, as $t \to \infty$, so that $(t^2 - 1)^{1/2} \to t$ and predominates over the logarithm, $z \to ht/\pi$ and $w = ct \to c\pi z/h$. But the assumed flow at infinity requires that on the z-plane $w \to Uz$. Hence $c = hU/\pi$, and

$$w = \phi + i\psi = \frac{hU}{\pi}\, t. \qquad [107c]$$

By substitution, z can easily be expressed in terms of w, but ϕ and ψ cannot be separated in terms of ordinary functions.

The relation between w and z is more conveniently studied in terms of real coordinates μ, ν on the z-plane defined by writing

$$t = \cosh(\mu + i\nu). \qquad [107d]$$

Using hyperbolic formulas listed in Section 32, and also separating real and imaginary parts in $z = x + iy$ and in w, it is found that

$$z = \frac{h}{\pi}\,[\mu + i\nu + \sinh(\mu + i\,\nu)], \qquad [107e]$$

$$x = \frac{h}{\pi}\,(\mu + \sinh\mu\cos\nu),\ y = \frac{h}{\pi}\,(\nu + \cosh\mu\sin\nu), \qquad [107f,g]$$

$$\phi = \frac{hU}{\pi}\cosh\mu\cos\nu,\ \ \psi = \frac{hU}{\pi}\sinh\mu\sin\nu). \qquad [107h,i]$$

The coordinates μ, ν are single valued within the region of interest provided $0 \leqq \mu$, $0 \leqq \nu \leqq \pi$. At infinity, $\mu \to \infty$, $\sinh\mu \to \cosh\mu$, $\mu/\sinh\mu \to 0$, and $\phi \to Ux$, as it must. The coordinate curves on which μ has a constant, positive value begin on the positive x-axis, where $\nu = 0$, cross the y-axis, and end on the line AB, on which $\nu = \pi$, $y = h$. The curve for $\mu = 0$ is the segment OB.

Some streamlines are shown in Figure 177. The point on OB at which $q = |U|$, or $\nu = \pi/2$, $y = (2 + \pi)h/2\pi = 0.818h$, is marked by a shortline.

The line $COBA$ is the streamline for $\psi = 0$. As the streamlines proceed from right to left, they all rise through a total distance h.

From Equation [107a], and Equation [107c] and $K = h/\pi$,

$$\left(\frac{dw}{dz}\right)^2 = \left(\frac{dw}{dt}\Big/\frac{dz}{dt}\right)^2 = U^2\,\frac{t-1}{t+1}$$

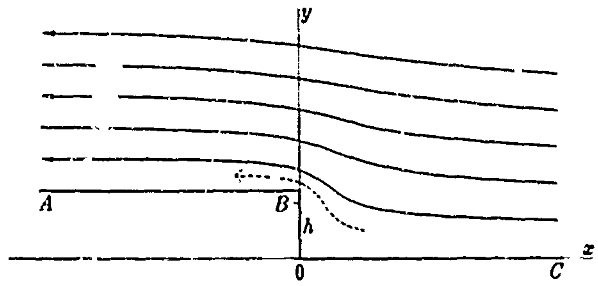


Figure 177 -- Streamlines for the situation in Figure 176. See Section 107.

and, from Equation [107d] and hyperbolic formulas listed in Section 32,

$$q^2 = \left|\frac{dw}{dz}\right|^2 = U^2\left[\frac{(\cosh\mu\cos\nu - 1)^2 + \sinh^2\mu\sin^2\nu}{(\cosh\mu\cos\nu + 1)^2 + \sinh^2\mu\sin^2\nu}\right]^{1/2}$$

or

$$q^2 = U^2\,\frac{\cosh\mu - \cos\nu}{\cosh\mu + \cos\nu}\ . \qquad [107j]$$

On the positive x-axis, $\mu \geqq 0$, $\nu = 0$, $q = |u|$ and

$$x = \frac{h}{\pi}(\sinh\mu + \mu), \quad u = -U\tanh\frac{\mu}{2}\ . \qquad [107k, l]$$

On the end OB of the obstacle, where $\mu = 0$, $0 \leqq \nu \leqq \pi$, $q = |v|$ and

$$y = \frac{h}{\pi}(\sin\nu + \nu);\ v = U\tan\frac{\nu}{2}\ . \qquad [107m, n]$$

On the face AB, $\mu \geqq 0$, $y = \nu = \pi$, $q = |u|$ and

$$x = -\frac{h}{\pi}(\sinh\mu - \mu),\ u = -U\coth\frac{\mu}{2}\ . \qquad [107o, p]$$

These velocities are most easily calculated thus: on OB, for example, $dy = (h/\pi)(\cos\nu + 1)d\nu$, $d\phi = -(hU/\pi)\sin\nu\,d\nu$, hence $v = -d\phi/dy = -U\sin\nu/(\cos\nu + 1) = -U\tan(\nu/2)$.

If the total force on OB is calculated by integrating the Bernoulli pressure and evaluating the improper integral in the usual way, the force is found to be zero provided the pressure in the undisturbed stream is zero. This result is correct, as will be shown in the next section. It is unsafe, however, to integrate the pressure up to a point at which $q \to \infty$; see Section 85.

(For notation and method; see Section 34; Reference 2, Section 10.6.)

108. STRAIGHT CHANNEL VARIED IN WIDTH.

Consider the flow along a two-dimensional channel of infinite length which has parallel straight sides but whose width undergoes a sudden change at one point from h_1 to h_2, as illustrated in Figure 178. Let the fluid approach from the right at uniform speed U. Then its velocity will ultimately become uniform again at the value $h_1 U/h_2$, since the same volume must pass all cross sections.

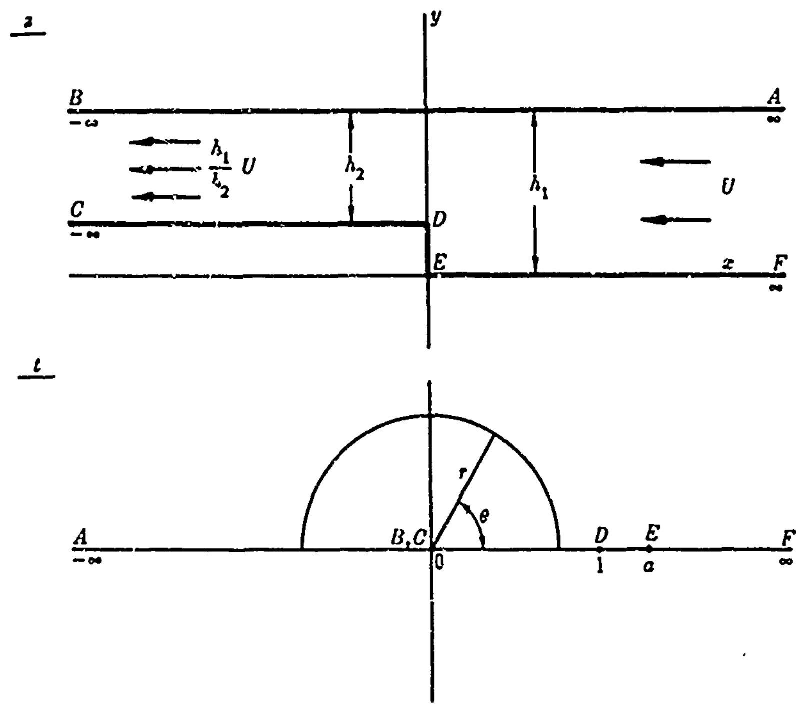


Figure 178 – Treatment of a straight channel abruptly varied in width. See Section 108.

The mathematical problem is an extension of that in the last section. The walls taken together in the order $ABCDEF$ as labeled in Figure 178 can be regarded as an infinite polygon with two vertices at infinity, AF and BC. At BC, where a change in direction of π occurs, let $t = 0$; at D and E, with exterior angles $-\pi/2$ and $\pi/2$, let $t = 1$ and $t = a > 1$, respectively. Then the Schwarz-Christoffel Equation [31a] becomes

$$\frac{dz}{dt} = \frac{K}{t}\left(\frac{t-1}{t-a}\right)^{1/2}, \qquad [108a]$$

Integrating by means of the substitution

$$\tau = \left(\frac{t-a}{t-1}\right)^{1/2} = \left(1 + \frac{a-1}{1-t}\right)^{1/2} = \left[a - \frac{(a-1)t}{t-1}\right]^{1/2}, \qquad [108b]$$

$$z = K\left(\ln\frac{1+\tau}{1-\tau} + \frac{1}{\sqrt{a}}\ln\frac{\sqrt{a}-\tau}{\sqrt{a}+\tau}\right) + L. \qquad [108c]$$

Here it can be assumed that the amplitudes of t, $t-1$, $t-a$, and hence also of $(t-a)/(t-1)$ range only from 0 to π, inclusively, those of τ, $1+\tau$ and $\sqrt{a}+\tau$ from 0 to $\pi/2$, and those of $1-\tau$ and $\sqrt{a}-\tau$ from $-\pi$ to 0.

As $t \to \pm\infty$, $\tau \to 1$; hence the real part of z becomes infinite. Thus $t = \infty$ at F and $t = -\infty$ at A, since t increases from A to B. Thus, on the t-plane the streamline AB becomes the negative real axis, the line $CDEF$, the positive real axis. Since the flow is toward BC, the transformed flow on the t-plane must be one of convergence toward $t = 0$. The simplest type of such convergent flow is that due to a line sink at $t = 0$, for which the complex potential may be written

$$w = c \ln t, \qquad [108d]$$

from Section 40, where $w = \phi + i\psi$ and c is a real constant.

To fix the constants, take the x-axis parallel to the channel. Then, to provide the assumed inflow at AF, it is necessary that at AF $w \to Uz$ + constant or $dw/dz \to U$; see Section 35. But

$$\frac{dw}{dz} = \frac{dw}{dt}\bigg/\frac{dz}{dt} = \frac{c}{K}\left(\frac{t-a}{t-1}\right)^{1/2}. \qquad [108e]$$

When $|t|$ is large, the last fraction becomes unity. Hence it is necessary that

$$\frac{c}{K} = U. \qquad [108f]$$

As BC or $t = 0$ is approached, dw/dz must reduce similarly to $h_1 U/h_2$. Hence, from Equation [108e],

$$\frac{h_1 U}{h_2} = \frac{c}{K}\sqrt{a}. \qquad [108g]$$

The fact that it is thus obviously possible to make the solution represent the assumed flow in distant parts of the channel confirms the choice of w as a function of t.

Furthermore, if t has a large negative real value, $1 < \tau < \sqrt{a}$; if t is large and positive, $0 < \tau < 1$. Hence, as t changes from a large negative to a large positive real value, all of the four binomials whose logarithms occur in Equation [108c] remain real, but $1 - \tau$ changes from negative to positive, and its amplitude from $-\pi$ to 0. The imaginary part of z is thereby increased by $-i\pi K$. But, on the diagram, z changes from A to F; hence the change in its imaginary part is also $-ih_1$. Thus $-i\pi K = -ih_1$. From this result and Equations [108f, g]

$$K = \frac{h_1}{\pi}, \quad c = \frac{h_1 U}{\pi}, \quad a = \frac{h_1^2}{h_2^2}. \qquad [108h, i, j]$$

Finally, at E or $t = a$, $\tau = 0$, $z = L$ by Equation [108c]. Hence, if the origin is placed at E, $L = 0$. Then, from Equations [108c], [108h, i, j], [108b] and [108d],

$$z = x + iy = \frac{1}{\pi}\left(h_1 \ln \frac{1+\tau}{1-\tau} + h_2 \ln \frac{h_1 - h_2\tau}{h_1 + h_2\tau}\right), \qquad [108k]$$

$$\tau = \left[\frac{h_2^2 e^{\pi w/(h_1 U)} - h_1^2}{h_2^2 (e^{\pi w/(h_1 U)} - 1)}\right]^{1/2}, \quad w = \phi + i\psi. \qquad [108l, m]$$

These equations fix ϕ and ψ as functions of x and y, but their interpretation is involved.

For the velocity, from Equations [108b, e] and Equations [108h, i],

$$q^2 = \left|\frac{dw}{dz}\right|^2 = U^2 \left|\frac{t-a}{t-1}\right| = U^2\, |\tau^2|. \qquad [108n]$$

On all walls parallel to U, τ is real and positive, so that, from Equation [108n], $\tau = q/U$. On AB, $t < 0$, $1 < \tau < \sqrt{a} = h_1/h_2$; on CD, $0 < t < 1$, $\tau > \sqrt{a} = h_1/h_2$; on EF, $t > a$, $\tau < 1$. Hence, from Equation [108n] and Equation [108k],

on AB:

$$U < q < h_1 U/h_2, \quad x = \frac{1}{\pi}\left(h_1 \ln \frac{q+U}{q-U} + h_2 \ln \frac{h_1 U - h_2 q}{h_1 U + h_2 q}\right),$$

on CD:

$$q > h_1 U/h_2, \quad x = \frac{1}{\pi}\left(h_1 \ln \frac{q+U}{q-U} + h_2 \ln \frac{h_2 q - h_1 U}{h_2 q + h_1 U}\right),$$

on EF:

$$q < U, \quad x = \frac{1}{\pi}\left(h_1 \ln \frac{U+q}{U-q} + h_2 \ln \frac{h_1 U - h_2 q}{h_1 U + h_2 q}\right).$$

On DE, $1 < t < a$, τ is imaginary and $\tau = iq/U$, and

$$y = \frac{2}{\pi}\left(h_1 \tan^{-1} \frac{q}{U} - h_2 \tan^{-1} \frac{h_2 q}{h_1 U}\right).$$

The force on DE due to the Bernoulli pressure in steady motion is most easily found from the conservation of momentum. Consider the fluid between two transverse planes drawn far away from DE and on opposite sides of it.

In a second, the net effect of the motion is the same as if a volume $h_1 U$ of this fluid were removed from the neighborhood of the rear plane (at the right) and inserted just ahead of the forward plane, gaining thereby momentum

$$\rho h_1 U \left(\frac{h_1}{h_2} U - U\right) = \rho h_1 U_1^2 \left(\frac{h_1}{h_2} - 1\right). \qquad [108o]$$

The momentum in the remainder of the space between the planes is unaltered. During the same time the difference of pressure between the two planes delivers momentum to the fluid of magnitude equal to the differential force multiplied by the time or

$$-\frac{1}{2} \rho U^2 h_1 - \left(-\frac{1}{2} \rho \frac{h_1^2}{h_2^2} U^2\right) h_2 = \frac{1}{2} \rho h_1 U^2 \left(\frac{h_1}{h_2} - 1\right). \qquad [108p]$$

The remainder of the gain in momentum must be furnished by a force due to negative pressure over DE; the reaction is a force of suction on DE, directed oppositely to the stream, of magnitude equal to Equation [108o] minus Equation [108p] or, per unit of length perpendicular to the flow,

$$F_1 = \frac{1}{2} \rho U^2 \frac{h_1}{h_2} (h_1 - h_2). \qquad [108q]$$

The force F_1 exceeds the force of suction on an equal area in the approaching stream by $\Delta F_1 = F_1 - \frac{1}{2} \rho U^2 (h_1 - h_2)$ or

$$\Delta F_1 = \frac{1}{2}\,\rho\, U^2\,(h_1 - h_2)\left(\frac{h_1}{h_2} - 1\right) = \frac{1}{2}\,\rho\, h_2\, U_2^2 \left(\frac{h_1 - h_2}{h_1}\right)^2 \qquad [108r]$$

in terms of the exit velocity $U_2 = h_1\, U/h_2$.

If $h_1 - h_2$ is held constant while $h_1 \to \infty$, the case of the last section is reproduced, and in this case $\Delta F_1 \to 0$. If, on the other hand, h_2/h_1 is very small, $\Delta F_1 = \rho\, h_2\, U_2^2/2$, approximately. This is the familiar suction force due to deficit of pressure on the walls of a vessel in the neighborhood of a relatively small orifice.

By putting together two flows of this type, one reversed as if by reflection in the direction of y, the flow can be represented through a plane-sided orifice of width $2h_2$, located in the plane end of a two-dimensional semi-infinite tank whose sides are $2h_1$ apart.

The velocity may be reversed at all points without affecting q or the geometrical flow net.

(For notation and method; see Section 34; Reference 2, Section 10.7.)

109. CHANNELS OF VARIOUS FORMS.

Channels with sides variously composed of straight lines, or in part curved, are described by Love[50] and Miyadzu;[176] for the introduction of a gate see Reference 177. Channels with curved walls are described by Sakurai.[252]

Branching channels are treated in Reference 2, Section 10.8; see also articles by Agostinelli,[178] Cisotti,[179] and Boverio.[180]

FREE STREAMLINES

110. NATURE OF FREE STREAMLINES.

Where a free surface occurs, as at the top of a mass of liquid or on the boundary of a region of cavitation, the usual requirement is uniformity of pressure. Problems involving this boundary condition are often difficult to solve.

If the motion is steady, however, and if gravity is absent, constancy of pressure is equivalent to constancy of q, the magnitude of the velocity, as is evident from the Bernoulli Equation [10d]. Furthermore, in steady motion the free surface is composed of streamlines. Thus in steady motion the boundary condition along a free streamline is that q is constant. This boundary condition is readily handled.

Alternatively, the space adjacent to the moving fluid, instead of being empty or filled with gas of negligible density, may be assumed to be filled with fluid of the same kind but at rest. The steady motion of the remaining fluid is not thereby affected, provided viscosity is entirely absent; it suffices to assume that the pressure in the stationary fluid or wake is the same as the constant pressure along the boundary surface between the two. At this boundary the velocity is then discontinuous, and the motion is there rotational; a sheet of vortices may be supposed to exist on the boundary. The general principles and theorems that hold for irrotational motion will hold within the moving fluid taken by itself, but they may not be applicable throughout a region that includes part of the surface of discontinuity.

The theoretical flow of a frictionless fluid past an obstacle as obtained on such an assumption shows more resemblance to the flow of an actual fluid than the theoretical flow in which the motion is everywhere continuous. In particular, a force is exerted on the obstacle. But in real cases a great deal of vortex motion is observed to exist in the wake.

In the mathematical theory of two-dimensional motion, constancy of the velocity q implies constancy of $|dw/dz|$ along a free streamline. It is found convenient to work with the variable

$$\zeta = -\frac{dz}{dw}\,, \tag{110a}$$

or, in terms of the velocity components u and v,

$$\zeta = -\left(\frac{dw}{dz}\right)^{-1} = -\frac{1}{-u + iv} = \frac{u + iv}{q^2}\,, \tag{110b}$$

where the last member is obtained by rationalizing the denominator and using $q^2 = u^2 + v^2$. Thus $|\zeta| = 1/q$.

Regarded as a function of z, $\zeta(z)$ effects a transformation from the z-plane onto a ζ-plane. Let this plane be drawn parallel to the z-plane and with the real axis of ζ parallel to the x-axis. Then the vector representing ζ has the same direction as the particle velocity at the corresponding point on the z-plane, since it makes with its real axis the angle $\tan^{-1}(v/u)$. Therein lies the special utility of the variable ζ.

Each streamline on the z-plane transforms into a curve on the ζ-plane, and this curve can be regarded as the corresponding streamline in a transformed motion. From the properties of ζ it is clear that any straight portion of a z streamline will transform into a segment of a parallel radius from the origin. As the z point traces a curved free streamline, on the other hand, along which q is constant, the ζ point traces the arc of a circle of radius $1/q$, centered at the origin.

Further transformations may then be made in terms of other variables until the problem is converted into a form in which the solution can be guessed.

In the alternative "hodograph method" of Prandtl, dw/dz, or $-1/\zeta$, is used as an auxiliary variable instead of ζ; see, for example, Betz and Petersohn.[181]

(For notation; see Section 34.)

111. EFFLUX FROM A TWO-DIMENSIONAL ORIFICE.

Consider the steady irrotational outflow of a liquid through a parallel-sided slot of width $2a$ in the infinite plane wall of a tank. Let gravity be assumed not to act. In Section 61 the problem was solved on the assumption that the liquid remains in contact with the outer face of the wall. It will now be assumed that the issuing liquid separates from the wall in the form of a jet bounded on each side by free streamlines.

On the z-plane let the slot be represented by the segment $(-a, 0)$ to $(a, 0)$ of the real axis, as in Figure 179; and, for simplicity, let the constant velocity along the free streamlines be $q = 1$.

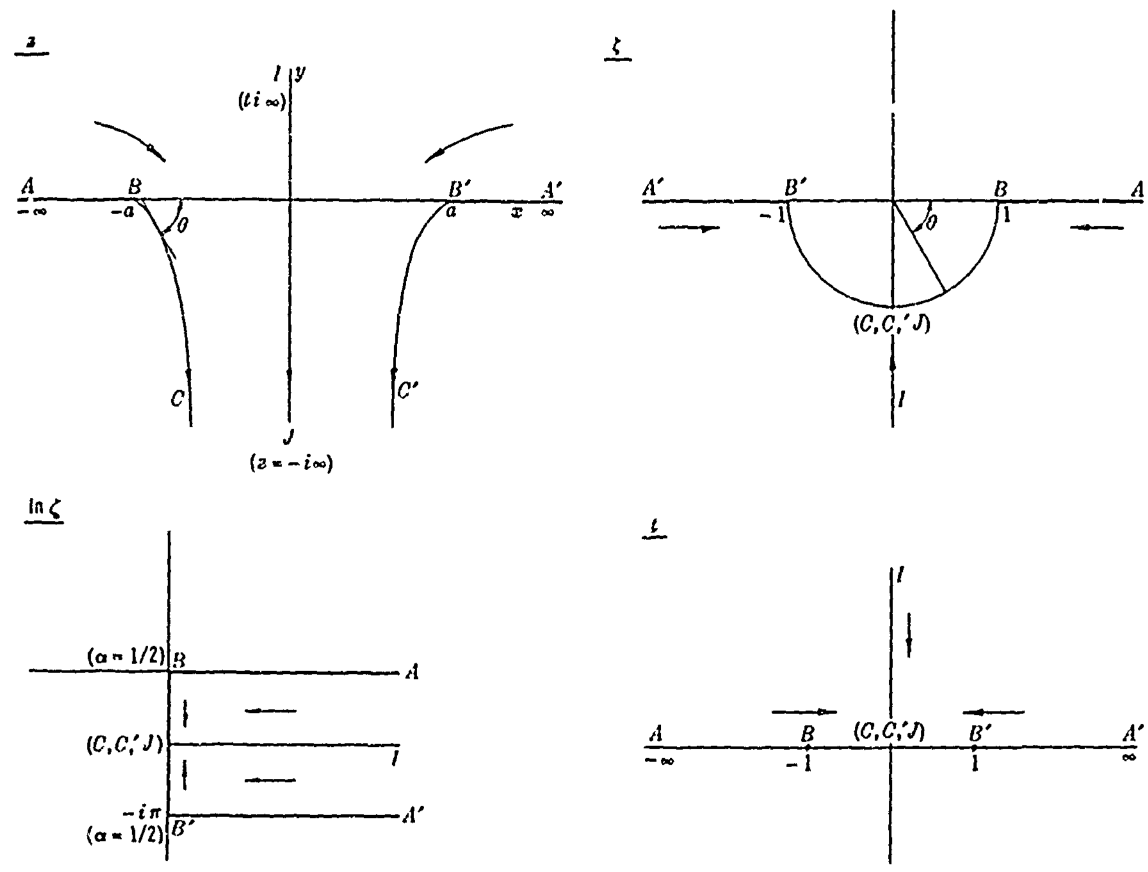


Figure 179 – Efflux from a two-dimensional orifice. See Section 111.

As explained in the last section, the motion is first studied in terms of the variable

$$\zeta = -\frac{dz}{dw} . \qquad [111a]$$

Along the wall AB the fluid velocity is directed toward B and its magnitude q increases from zero at A, or at infinity, to unity at B. Hence the ζ vector, of magnitude $1/q$, lies on its positive real axis, and its end moves on the ζ-plane from ∞ when z is at A to $\zeta = 1$ at B. Along the curving free streamline BC, ζ then moves along the unit circle below its real axis; the direction of the ζ vector is at each point that of the tangent to BC. A similar streamline coming from the right transforms into the negative real axis of ζ from $-\infty$ to -1, together with another part of the unit circle below the axis. Finally, because of the obvious symmetry, there must be a central streamline IJ which is straight throughout and becomes part of the imaginary axis of ζ.

The boundary $ABB'A'$ thus traced on the ζ-plane, consisting of two segments of the real axis and a semicircle, is next transformed into the entire real axis of another variable t. For this purpose a transformation is first made to a new variable

$$\ln \zeta = \ln |\zeta| + i \operatorname{amp} \zeta \qquad [111b]$$

where $\ln |\zeta|$ stands for the ordinary real logarithm. This converts the boundary into a semi-infinite rectangle. AB becomes the positive real axis of $\ln \zeta$. On the unit circle, $\ln |\zeta| = 0$ and amp ζ ranges from 0 to $-\pi$; hence the semicircle below the real axis of ζ becomes the segment of the imaginary axis from $\ln \zeta = 0$ to $\ln \zeta = -i\pi$. On $A'B'$, amp $\zeta = -i\pi$, hence, on the plane of $\ln \zeta$, $A'B'$ becomes a line parallel to AB at a distance $i\pi$ below it.

The Schwarz-Christoffel transformation, discussed in Section 31, is now used to convert the rectangle $ABB'A'$ on the plane of $\ln \zeta$ into the real axis of t. The space between AB and $A'B'$ is to be regarded as the interior of the rectangle, since, as the boundary $ABCJC'B'A'$ on the z-plane is traced, the fluid lies on the left. Hence exterior angles of $\pi/2$ occur at B and B'. Let the corresponding values of t be chosen as -1 and $+1$.

Then, putting $a_1 = -1$, $\alpha_1 = 1/2$, $a_2 = 1$, $\alpha_2 = 1/2$ in Equation [31a],

$$\frac{d}{dt} \ln \zeta = K(t+1)^{-1/2} (t-1)^{-1/2} \qquad [111c]$$

and, integrating,

$$\ln \zeta = K \ln [t + (t^2 - 1)^{1/2}] + L. \qquad [111d]$$

The choice of amplitudes here is as in Section 107.

The constants K and L can now be adjusted to bring the corners B and B' into the correct position on the plane of $\ln \zeta$. Inserting into Equation [111d] $t = -1$ for $\ln \zeta = 0$, then $t = 1$ for $\ln \zeta = -i\pi$:

$$0 = K \ln(-1) + L = i\pi K + L, \; -i\pi = L,$$

whence $K = 1$ and

$$\ln \zeta = \ln [t + (t^2 - 1)^{1/2}] - i\pi = \cosh^{-1} t - i\pi, \qquad [111e]$$

$$\zeta = -t - (t^2 - 1)^{1/2}, \quad t = -\frac{1}{2}(\zeta + \zeta^{-1}), \qquad [111f, g]$$

since $e^{-i\pi} = -1$. Here, when t is real and negative, $(t^2 - 1)^{1/2} = -\sqrt{t^2 - 1}$.

Now, as z traces one of the streamlines ABC or $A'B'C'$, t moves along its real axis from $-\infty$, or from $+\infty$, toward 0; and ψ obviously has different values on these two streamlines, which bound a jet of fluid. Hence the flow on the t-plane resembles that toward a sink located at the origin. This fact suggests the following trial assumption as to the complex potential:

$$w = c \ln t, \; t = e^{w/c},$$

where c is real; see Section 40. According to this assumption, the free streamlines extend up to the point $t = 0$, or $\zeta = -i$.

Then $dw = c\,dt/t$ and, from Equation [111a] and Equation [111f], integrating,

$$dz = -\zeta\, dw = c\,[t + (t^2 - 1)^{1/2}]\, dt/t;$$

$$z = c\left[t + (t^2 - 1)^{1/2} + \sin^{-1}\frac{1}{t}\right] + k. \qquad [111h]$$

For the significance of $(t^2 - 1)^{1/2}$ see Section 107.

To evaluate $\sin^{-1}$, consider, in general, $\sin^{-1} z$. Write

$$\sin^{-1} z = \nu + i\xi,$$

where ν and ξ are real. Then $z = x + iy = \sin(\nu + i\xi) = \sin\nu \cosh\xi + i\cos\nu \sinh\xi$ and

$$x = \sin\nu \cosh\xi, \quad y = \cos\nu \sinh\xi. \qquad [111i, j]$$

Thus ξ and ν serve as elliptic coordinates on the z-plane and can be found for any point; see Section 82, where $\eta = (\pi/2) = \nu$.

In analogy with Equations [82e, f],

$$\cosh\xi = \frac{1}{2}\{[(x + 1)^2 + y^2]^{1/2} + [(x - 1)^2 + y^2]^{1/2}\}, \qquad [111k]$$

$$\sin \nu = \frac{1}{2} \{[(x+1)^2 + y^2]^{1/2} - [(x-1)^2 + y^2]^{1/2}\}. \qquad [111l]$$

On the real axis of z, where $|x| \leq 1$, $\xi = 0$ and $\sin^{-1} z = \sin^{-1} x = \nu$ and has its ordinary meaning; where $|x| \geqq 1$, $\cos \nu = 0$, $x = \pm \cosh \xi$, and ξ may have either sign. so that if $x > 1$,

$$\sin^{-1} z = \sin^{-1} x = \frac{\pi}{2} \pm i \cosh^{-1} x;$$

if $x < -1$,

$$\sin^{-1} z = \sin^{-1} x = -\frac{\pi}{2} \pm i \cosh^{-1} (-x),$$

where either sign may be chosen.

The variables ν and ξ are doubly many valued, hence so is $\sin^{-1} z$; as with real angles, if one value is $\sin^{-1} z = u$, then others are $u \pm 2\pi n$ where n is any integer, and also $(1 \pm 2n)\pi - u$. This complication can be removed only at the expense of introducing discontinuities. A convenient range for ν may be $-\frac{\pi}{2} \leqq \nu \leqq \pi/2$. Using this range, $\sin^{-1} z$ is discontinuous along the x-axis where $|x| = \cosh \xi > 1$; there the sign of ξ is indeterminate, whereas elsewhere ξ has the sign of y. With this convention, Figure 127 may be used by assuming that on the plot $c = 1$ and $\eta = (\pi/2) - \nu$.

In the present problem the values of $1/t$ lie on or below the real axis. For such values continuity can be preserved with use of the ranges $\xi \leqq 0$, $-\pi/2 \leqq \nu \leqq \pi/2$, provided it is agreed that, for a real number x, in terms of the positive $\cosh^{-1} x$,

if $x > 1$,

$$\sin^{-1} x = \frac{\pi}{2} - i \cosh^{-1} x; \qquad [111m]$$

if $x < -1$,

$$\sin^{-1} x = -\frac{\pi}{2} - i \cosh^{-1} x. \qquad [111n]$$

At B', or at $t = 1$, $z = a$; at B, or $t = -1$, $z = -a$. Inserting these values in Equation [111h] and evaluating c and k, it is found that

$$c = \frac{2a}{2+\pi}, \quad w = \frac{2a}{2+\pi} \ln t, \qquad [111o,p]$$

$$z = \frac{2a}{2+\pi} \left[t + (t^2 - 1)^{1/2} + \sin^{-1} \frac{1}{t} \right]. \qquad [111q]$$

Since $w = \phi + i\psi$, ϕ and ψ are now fixed, but they cannot be expressed in terms of z by means of ordinary functions.

The velocity of the fluid at any point is given by

$$-u + iv = \frac{dw}{dz} = -\frac{1}{\zeta} = [t + (t^2 - 1)^{1/2}]^{-1}. \qquad [111r]$$

On the free streamlines $q = 1$. On the side $A'B'$ of the wall of the tank, t is real and $t > 1$; hence

$$q = -u = [t + (t^2 - 1)^{1/2}]^{-1}, \quad t = \frac{1}{2}\left(\frac{1}{q} + q\right), \quad (t^2 - 1)^{1/2} = \frac{1}{2}\left(\frac{1}{q} - q\right).$$

Also, z is real and $z = x$. Thus, from Equation [111q],

$$x = \frac{2a}{2 + \pi}\left[\frac{1}{q} + \sin^{-1}\left(\frac{2q}{q^2 + 1}\right)\right] \qquad [111s]$$

where $\sin^{-1}$ is in the first quadrant. The side $A'B'$ is symmetrical to AB.

Along the median plane, or IJ in Figure 179,

$$t = i\,|t|, \quad (t^2 - 1)^{1/2} = i\,(|t|^2 + 1)^{1/2}.$$

Thus

$$q = -v = [|t| + (|t|^2 + 1)^{1/2}]^{-1}$$

and

$$|t| = \left(\frac{1}{q} - q\right)\Big/2, \quad (|t|^2 + 1)^{1/2} = \left(\frac{1}{q} + q\right)\Big/2.$$

Also, if $s = \sin^{-1}(1/t) = \sin^{-1}(1/i\,|t|)$, then $1/|t| = i \sin s = \sinh(is)$, hence $s = -i \sinh^{-1}(1/|t|)$. Finally, $z = iy$. Hence, from Equation [111q].

$$y = \frac{2a}{2 + \pi}\left[\frac{1}{q} - \sinh^{-1}\frac{2q}{1 - q^2}\right]. \qquad [111t]$$

In describing the form of the free streamlines, a more convenient parameter than t is the angle θ, actually negative, between the x-axis and the tangent to the streamline; and a fresh integration is less troublesome, because of the singularity at $t = 0$. On a free streamline

$$\zeta = e^{i\theta}, \quad \cos\theta = \frac{1}{2}(\zeta + \zeta^{-1}) = -t. \qquad [111u, v]$$

Hence, using Equations [111a] and [111p],

$$dz = dx + idy = -\zeta\, dw = \frac{2a}{2 + \pi}(\cos\theta + i\sin\theta)\frac{\sin\theta}{\cos\theta}\, d\theta.$$

Separating dx and dy and integrating along the left-hand streamline,

$$x = - \frac{a}{2 + \pi} (\pi + 2 \cos \theta), \tag{111w}$$

$$y = \frac{2a}{2 + \pi} \left[\ln \tan \left(\frac{1}{4} \pi + \frac{1}{2} \theta \right) - \sin \theta \right]. \tag{111x}$$

Here $-\frac{\pi}{2} \leqq \theta < 0$. The right-hand streamline is then the mirror image in the y-axis of this one.

The issuing jet eventually becomes straight; its limiting width is twice the value of $|x|$ when $\theta = -\pi/2$ or $2\pi a/(2 + \pi)$. The ratio of contraction, relative to the width of the orifice or $2a$, is thus

$$\frac{\pi}{2 + \pi} = 0.611.$$

Since the velocity in the ultimate jet is uniform, the volume of fluid that issues per second, per unit length, is $2\pi a/(2 + \pi)$.

Thus on the z-plane the streamlines do not converge at infinity, as is indicated by the distinct labeling C, J, C' in Figure 179, but on the ζ-, $\ln \zeta$-, and t-planes they converge to a finite point, as at $\zeta = -i$ or $t = 0$.

At $\theta = 0$, or the edge of the orifice, $dy/dx = \sin^2 \theta / \cos^2 \theta = 0$, but $d\theta/dx \propto 1/\sin \theta \to \infty$. Thus, although there is no discontinuity in the slope of the streamline, its curvature at the edge is infinite.

The free streamlines are plotted to scale in Figure 179, and an enlarged plot of one half of the symmetrical diagram is shown in Figure 180.

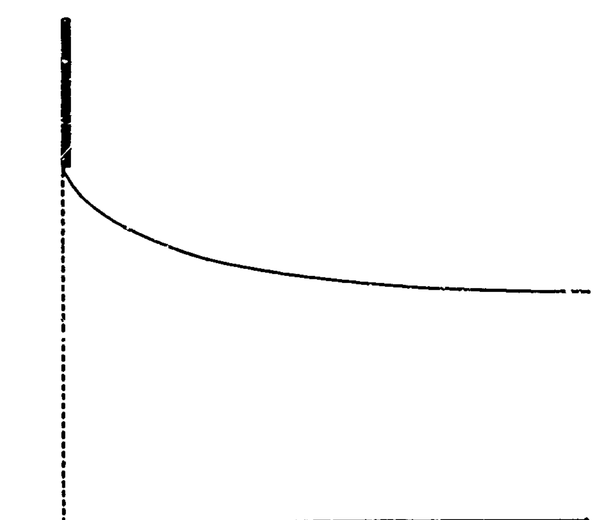


Figure 180 – Efflux as in Figure 179: one side of the issuing jet in more detail. (Copied from Reference 1.)

If the velocity along the free streamlines is made U instead of unity, the only effect is to replace Equation [111p] by $w = 2aU \ln t/(2 + \pi)$, and to multiply all velocities by U; q is then replaced in all formulas by q/U. The flow pattern is unaltered. U may be negative, so that the fluid is entering through the slot.

If p_∞ is the pressure at infinity in the mass of fluid where $q = 0$, and p_f the pressure at the surface of the jet itself, then, from the Bernoulli equation

$$p_\infty - p_f = \frac{1}{2} \rho U^2.$$

This equation fixes U^2 when $p_\infty - p_f$ is given.

In the presence of gravity the velocity is not uniform along the free surface and the problem is more difficult.

(For notation and method; see Section 34; Reference 1, Article 75, where a is replaced by $(\pi + 2)\, b/\pi$; Reference 2, Section 11.53.)

112. TWO-DIMENSIONAL BORDA'S MOUTHPIECE.

The transformations in the last section are easily modified so as to allow the plane boundaries to be inclined to each other at any angle, with preservation of the symmetry. The integration is simple if the planes are made parallel, so as to form a parallel-sided mouthpiece enclosing the issuing jet. Let the x-axis be rotated so as to lie in the plane of symmetry, with the edges of the sides at $(0, a)$, $(0, -a)$, as in Figure 181.

On the ζ-plane, the boundaries AB, $A'B'$ now coincide and lie on the positive real axis; in the figure they are drawn slightly separated for clarity. If amp $\zeta = 0$ on AB, amp $\zeta = -2\pi$ on $A'B'$; for, on the z-plane the direction of the velocity rotates through a clockwise angle of 360 deg in passing from AB through the fluid to $A'B'$. The t diagram is the same as before, and Equation [111d] holds again. Substituting in it, first $\ln \zeta = 0$ when $t = -1$, then $\ln \zeta = -2\pi i$ when $t = 1$, noting that here $\ln(-1) = \pi i$, $\ln(0) = 0$, and determining K and L, Equation [111d] becomes

$$\ln \zeta = 2 \ln [t + (t^2 - 1)^{1/2}] - 2\pi i,$$

whence, since $e^{-2\pi i} = 1$,

$$\zeta = [t + (t^2 - 1)^{1/2}]^2. \qquad [112a]$$

As before,

$$w = \phi + i\psi = c \ln t \qquad [112b]$$

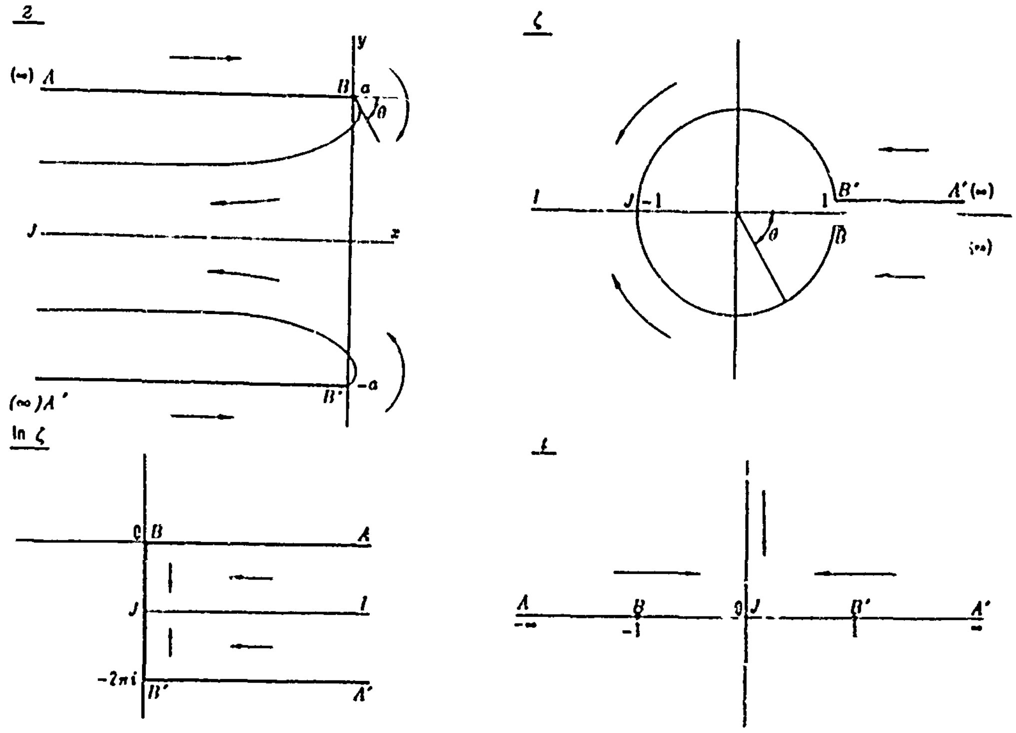


Figure 181 – The two-dimensional Borda's mouthpiece. See Section 112.

where c is real. Hence, using Equations [111a] and [112a],

$$dz = -\zeta dw = -c[2t^2 - 1 + 2t(t^2-1)^{1/2}]\, dt/t,$$

$$z = -c\{t^2 + t(t^2-1)^{1/2} - \ln t - \ln[t + (t^2-1)^{1/2}]\} + k. \qquad [112c]$$

Here the amplitudes of the algebraic functions of t, except t and $t^2 - 1$, are confined to the range from 0 to π. At B or $z = ia$, $t = -1$; at $z = -ia$, $t = 1$. Hence,

$$c = \frac{a}{\pi}, \quad k = \frac{a}{\pi} - ia. \qquad [112d,e]$$

The velocity of the fluid at any point, using Equation [112a], is given by

$$-u + iv = \frac{dw}{dz} = -\frac{1}{\zeta} = -[t + (t^2-1)^{1/2}]^{-2}. \qquad [112f]$$

By proceeding as in the last section it is found that along the walls AB and $A'B'$ $q = u$ and

$$x = -\frac{a}{\pi}\left[\frac{1-q}{2q} - \ln\frac{1+q}{2q}\right], \qquad [112g]$$

whereas, along the median plane or the x-axis, $q = -u$ and

$$x = \frac{a}{\pi}\left[\frac{1+q}{2q} + \ln\frac{1-q}{2q}\right]. \qquad [112h]$$

To trace the upper free streamline, to which the lower is symmetrical, introduce again the angle θ, which is here negative. Then

$$\zeta^{1/2} = e^{i\theta/2} = \cos\frac{\theta}{2} + i\sin\frac{\theta}{2}, \quad \cos\frac{\theta}{2} = \frac{1}{2}(\zeta^{1/2} + \zeta^{-1/2}) = -t,$$

$$dz = dx + idy = -\zeta dw = c\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)^2 \tan\frac{\theta}{2}\,\frac{d\theta}{2},$$

$$x = \frac{a}{\pi}\left(\sin^2\frac{\theta}{2} - \ln\sec\frac{\theta}{2}\right), \quad y = \frac{a}{2\pi}(\theta - \sin\theta) + a, \qquad [112i,j]$$

after integrating and using Equation [112d]. As $\theta \to -\pi$, $x \to -\infty$ and $y \to a/2$. Thus the entire jet is ultimately only $2a - a$ or a wide, and the ratio of contraction is $a/2a = 1/2$. The volume of fluid issuing per second and per unit length is a.

One side of the jet is plotted in Figure 182.

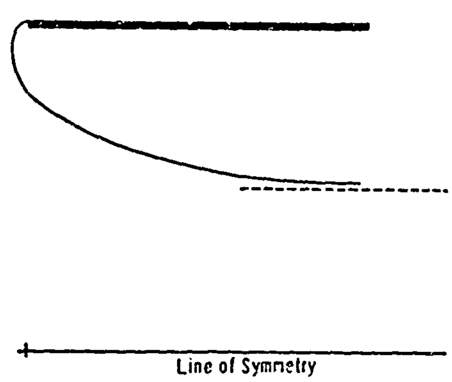


Figure 182 – Borda's mouthpiece: one side of the jet. (Copied from Reference 1.)

The same remarks hold here as in the last section in regard to the terminal form of the free streamlines, the pressure, the velocity and the effect of gravity.

(For notation and method see Section 34; Reference 1, Article 74, where $b = a/2$; Reference 2, Section 11.51.)

113. INFINITELY WIDE STREAM INCIDENT NORMALLY ON A PLANE LAMINA.

Let a steady uniform stream impinge perpendicularly on a fixed plane lamina of breadth l and continue beyond it with two free boundaries; see Figure 183.

From the symmetry, there will be a central streamline which is straight until it meets the lamina at its center C, then divides and follows the lamina to its edges, from which each half continues as a free streamline AI or $A'I'$, on which the constant velocity will be assumed to be unity. The general method described in Section 110 is applicable, and the mathematical

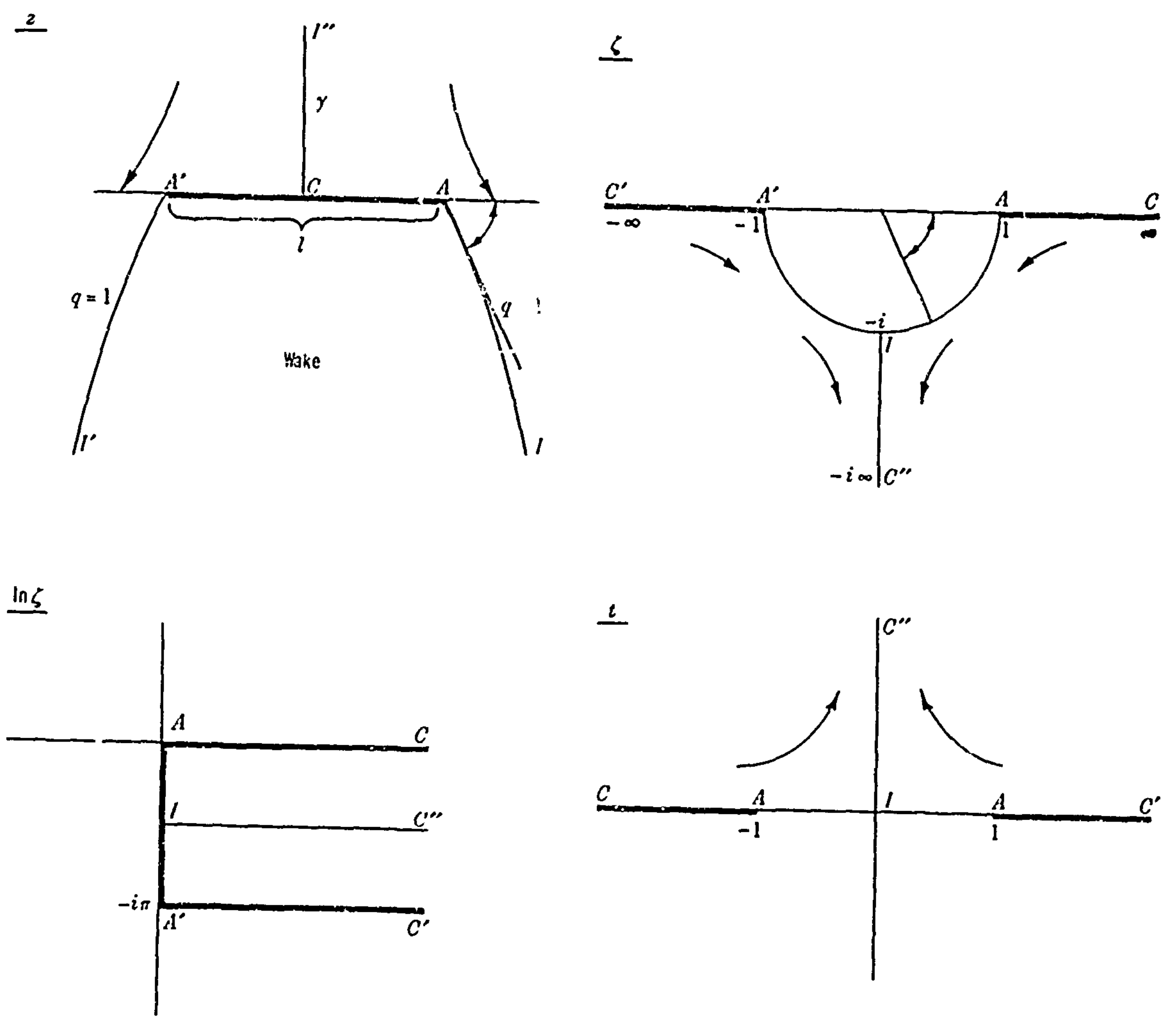


Figure 183 – Plate in a stream with wake behind it. See Section 113.

treatment of Section 111 requires only certain changes in order to fit the present case. The point C is a stagnation point, from which the fluid flows away toward both sides. Hence the parts CA and CA' of the lamina are represented by segments of the real axis of ζ, as shown in Figure 183, while the free streamlines follow the unit circle below the real axis as before. The geometrical boundary on the ζ-plane is thus the same as in Section 111, and the same transformation from ζ to t can be used:

$$\zeta = -\frac{dz}{dw} = -t-(t^2-1)^{1/2}, \quad t = -\frac{1}{2}\left(\zeta + \frac{1}{\zeta}\right). \qquad [113a, b]$$

For the amplitudes, see Section 107.

The flow on the t-plane is again along the real axis toward the origin, but in the present case ψ has the same value on the two halves of the axis, which represent on the z-plane parts of a single divided streamline. Hence there cannot be a source or sink at the origin; the fluid must flow away along the imaginary axis. A simple flow of this type is that of Section 38, whose complex potential may be written

$$w = -\frac{c}{t^2} \qquad [113c]$$

In this flow the axial streamlines continue to the origin; on the ζ-plane, therefore, the corresponding curves continue to the point I or $\zeta = -i$. It follows that the free streamlines become parallel at infinity.

From Equation [113a], $dz = -\zeta dw$. After substituting from Equations [113a] and [113c], integrating with the help of the substitution $t = 1/u$ and choosing c and the constant of integration so as to make $z = \pm l/2$ at A and A' or $t = \mp 1$, it is found that

$$c = \frac{l}{\pi + 4}, \qquad [113d]$$

$$z = -\frac{2l}{\pi+4}\left[\frac{1}{t} + \frac{1}{2t^2}(t^2-1)^{1/2} + \frac{1}{2}\sin^{-1}\frac{1}{t}\right]. \qquad [113e]$$

Here, for real t, and $|t| \geq 1$, $-\pi/2 \leq \sin^{-1}\frac{1}{t} \leq \pi/2$.

The particle velocity at any point is obtainable from the equation

$$-u + iv = \frac{dw}{dz} = -\frac{1}{\zeta} = t - (t^2-1)^{1/2}. \qquad [113f]$$

Clearly $|z| \to \infty$ only as $t \to 0$, or as $\zeta \to -i$; then, also, $u \to 0$, $v \to -1$. Thus, in particular, the velocity at infinity is -1 in the approaching stream. Mathematically, the point I or $\zeta = -i$ corresponds to all directions of recession to infinity on the z-plane, such as those labeled I, I', I'' in Figure 183. In the same way, C, C', and C'' on the ζ-plane all correspond to the single point C on the z-plane. At these points there is a failure of one or more of the conformal transformations.

By proceeding as in Section 111 it is found that on $A'A$

$$x = \pm \frac{2l}{\pi + 4} \left(\frac{q(3 + q^2)}{(1 + q^2)^2} + \frac{1}{2} \sin^{-1} \frac{2q}{1 + q^2} \right), \qquad [113g]$$

whereas along the median streamline $I''C$, or the positive y-axis,

$$y = \frac{2l}{\pi + 4} \left(\frac{q(3 - q^2)}{(1 - q^2)^2} + \frac{1}{2} \sinh^{-1} \frac{2q}{1 - q^2} \right). \qquad [113h]$$

Along the right-hand free streamline, proceeding as in Section 111, where Equations [111u, v] still hold but w is now given by Equation [113c] and c by Equation [113d],

$$dx + i\,dy = -\zeta\,dw = \frac{2l}{\pi + 4} (\cos\theta + i \sin\theta) \frac{\sin\theta}{\cos^3\theta}\, d\theta,$$

$$x = \frac{2l}{\pi + 4} \left(\sec\theta + \frac{\pi}{4} \right), \quad y = \frac{l}{\pi + 4} \left[\sec\theta \tan\theta - \ln \tan \left(\frac{\theta}{2} + \frac{\pi}{4} \right) \right], \qquad [113i]$$

after integrating and choosing the constants of integration to make $x = l/2$, $y = 0$ at A, where $\theta = 0$. The other free streamline is symmetrical with this one.

As $\theta \to -\pi/2$, which is the lower limit for θ, $x \to \infty$ and $y \to -\infty$. Thus the free streamlines eventually approach parallelism to the direction of incidence. They also become parallel to each other, but so slowly that the separation between them continues to increase without limit. A larger plot of one free streamline is shown in Figure 184, and a few streamlines near the lamina are shown in Figure 185.

The *pressure* on the free surface must be the same as the pressure in the fluid at infinity, since the velocity in both locations is unity. On the upper surface of the lamina it is higher. The total net force on the lamina in the direction of the stream per unit of its length is, from the Bernoulli equation,

$$F_1 = \frac{1}{2} \rho \int_{-l/2}^{l/2} (1 - q^2)\, dx. \qquad [113j]$$

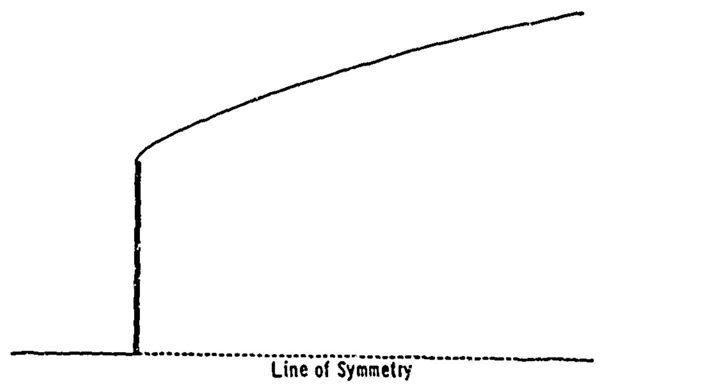


Figure 184 – Plate in a stream: one of the free streamlines. (Copied from Reference 1.)

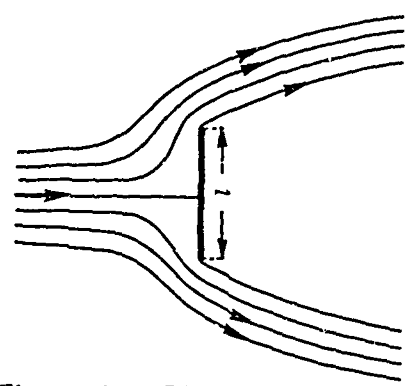


Figure 185 – Plate in a stream: a few streamlines. (Copied from Reference 8.)

On the lamina the variables are real, so that

$$dx = dz = -\zeta dw = -2c\zeta \frac{dt}{t^3}\ , \quad q^2 = \left|\frac{dw}{dz}\right|^2 = \frac{1}{\zeta^2}\ ,$$

$$F_1 = -\rho c \left(\int_{-\infty}^{-1} + \int_{1}^{\infty}\right)\left(\zeta - \frac{1}{\zeta}\right)\frac{dt}{t^3} = 4\rho c \int_1^{\infty} \sqrt{t^2 - 1}\ \frac{dt}{t^3} \qquad [113k]$$

and

$$F_1 = \frac{\pi \rho l}{\pi + 4} \qquad [113l]$$

using Equation [113a] and Equation [113d]; the integrals can be simplified by substituting $t = 1/u$, and the integral from $-\infty$ to -1 equals that from 1 to $+\infty$ by symmetry.

In the more general case in which the velocity at infinity is U, the functions w, ϕ, ψ and all velocities are multiplied by U, but the geometry of the flow net is unaltered. The force is multiplied by U^2.

For a solution in which the pressure on the free surface is different from that at infinity; see Reference 7, page 134.

(For notation and method; see Section 34; Reference 1, Article 76; Reference 2, Section 12.20.)

114. INFINITE STREAM OBLIQUE TO A PLANE LAMINA.

The solution of the last section is easily modified so as to allow the stream to impinge at an angle α with the plane of the lamina, as illustrated in Figure 186. Let α be measured from the positive x-axis. As before, let the velocity on the free streamlines and at infinity be unity.

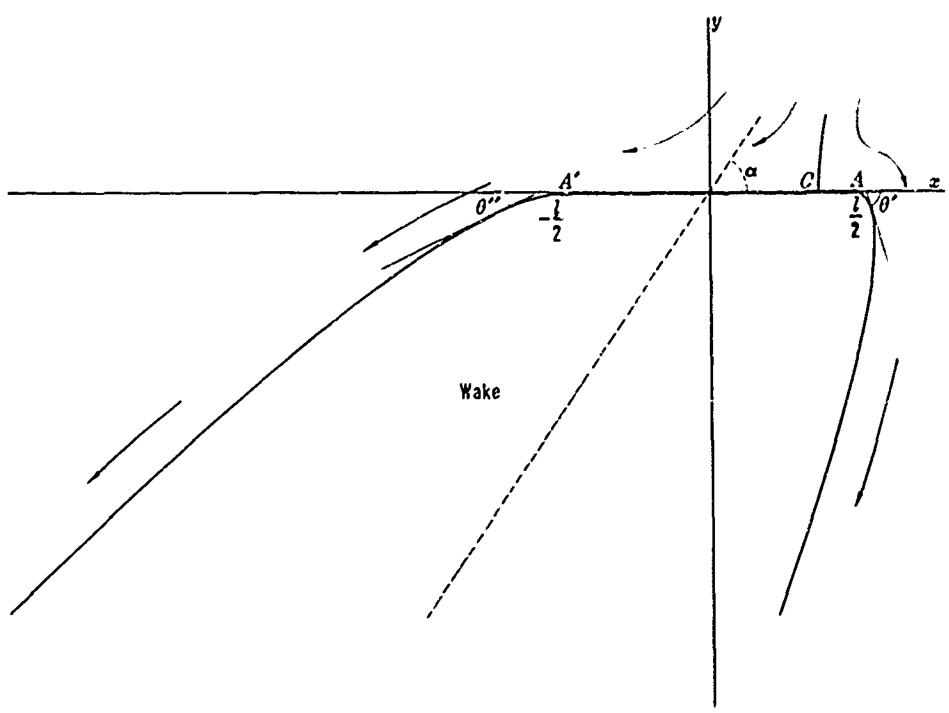


Figure 186 – Plate in a stream incident obliquely, with a wake behind it. See Section 114.

The dividing streamline will now meet the lamina perpendicularly at a stagnation point C that is displaced from the center. The transformations from ζ to $\ln \zeta$ and t are unaltered; but, at the point corresponding to infinity in the approaching stream, the vector representing ζ, having the direction of the velocity, must lie on the radius, $\theta = \alpha - \pi$, and the corresponding value of t will lie to the right of the origin, as shown in Figure 187. When $\zeta = e^{i(\alpha - \pi)} = -\cos\alpha - i\sin\alpha$, then $t = \cos\alpha$ by Equation [113b]. The flow on the t-plane is easily displaced so as to transfer its central point from the origin to $t = \cos\alpha$; in place of Equation [113c], let

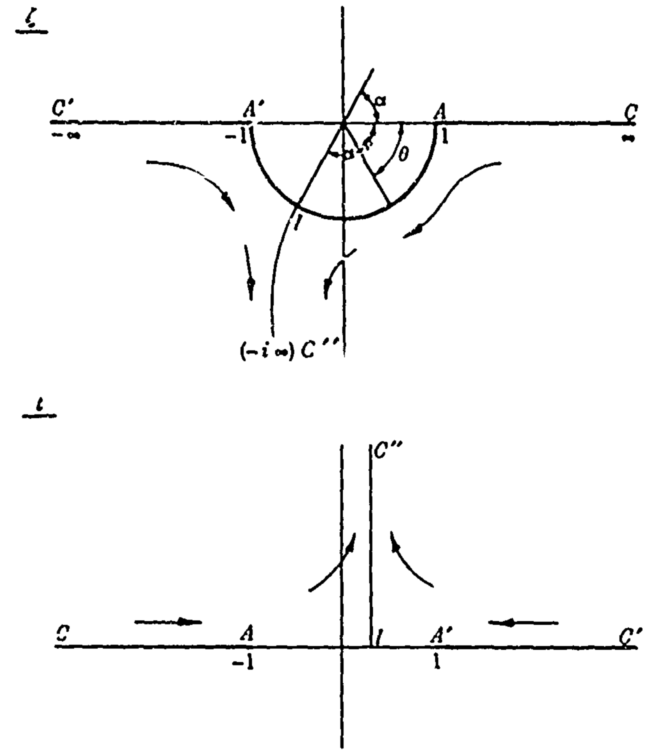


Figure 187 – Diagram for oblique incidence on a plate. See Section 114.

$$w = -\frac{c}{(t - \cos\alpha)^2} \quad . \tag{114a}$$

Then, from Equations [113a] and [114a],

$$dz = -\zeta\, dw = (t + \sqrt{t^2 - 1})\,\frac{2c\, dt}{(t - \cos\alpha)^3} \quad . \tag{114b}$$

After integrating, by means of the substitution $u = (t - \cos\alpha)^{-1}$, and choosing c and the constant of integration to make $z = \pm\, l/2$ for $t = \mp 1$, it is found that

$$c = \frac{l \sin^4\alpha}{4 + \pi \sin\alpha} \quad , \tag{114c}$$

$$z = \frac{l}{4 + \pi\sin\alpha}\left[\frac{\cos\alpha - 2t}{(t - \cos\alpha)^2}\sin^4\alpha + \frac{t\cos\alpha - 1}{(t - \cos\alpha)^2}\,(t^2 - 1)^{1/2}\sin^2\alpha \right.$$

$$\left. + \sin\alpha\,\sin^{-1}\frac{t\cos\alpha - 1}{t - \cos\alpha} + \cos\alpha\,(3 - \cos^2\alpha)\right]. \tag{114d}$$

Here, for real t and $|t| \geqq 1$, $(t^2 - 1)^{1/2}$ has the sign of t, and $-\pi/2 \leq \sin^{-1} \dfrac{1}{t} \leq \pi/2$.

For the particle velocity, Equation [113f] still holds.

On the lamina itself, t is real and $|t| > 1$, also $q = \pm u$, hence, from Equation [113f],

$$t = \mp \frac{1 + q^2}{2q} \quad , (t^2 - 1)^{1/2} = \mp \frac{1 - q^2}{2q} \quad ; \qquad [114e,f]$$

the sign is negative where $x > x_c$, positive where $x < x_c$, where x_c is the coordinate of C. These equations and Equation [114d], in which now $z = x$, connect q with x. The distance of C from the center 0 is the value of z at $|t| = \infty$ or

$$x_c = \frac{l}{4 + \pi \sin \alpha} \left[2 \cos \alpha \, (1 + \sin^2 \alpha) + \left(\frac{\pi}{2} - \alpha \right) \sin \alpha \right] , \qquad [114g]$$

where α is in radians.

The treatment of the free streamlines is nearly the same as that in Section 113. Taking w from Equation [114a] instead of Equation [113c],

$$dz = dx + i\,dy = 2c\,(\cos\theta + i \sin\theta) \frac{\sin\theta \, d\theta}{(\cos\theta + \cos\alpha)^3} \; .$$

After separating real and imaginary parts, x and y are found by integrating and adjusting the constant of integration. The formulas are most conveniently written in terms of distances and angles measured positively with the stream, or $y' = -y$, $\theta' = -\theta$, $\theta'' = \theta + \pi = \pi - \theta'$.

Right-hand streamline, $0 \leqq \theta' < \pi - \alpha$:

$$x = \frac{l}{2} + \frac{l \sin^4 \alpha}{4 + \pi \sin \alpha} \left(\frac{2 \cos \theta' + \cos \alpha}{(\cos \theta' + \cos \alpha)^2} - \frac{2 + \cos \alpha}{(1 + \cos \alpha)^2} \right) , \qquad [114h]$$

$$y' = \frac{l \sin \alpha}{4 + \pi \sin \alpha} \left[\frac{\sin \alpha \sin \theta' (1 + \cos \alpha \cos \theta')}{(\cos \theta' + \cos \alpha)^2} - \ln \frac{1 + \cos(\alpha - \theta')}{\cos \theta' + \cos \alpha} \right] . \qquad [114i]$$

Left-hand streamline, $0 \leqq \theta'' < \alpha$:

$$x = -\frac{l}{2} - \frac{l \sin^4 \alpha}{4 + \pi \sin \alpha} \left(\frac{2 \cos \theta'' - \cos \alpha}{(\cos \theta'' - \cos \alpha)^2} - \frac{2 - \cos \alpha}{(1 - \cos \alpha)^2} \right) , \qquad [114j]$$

$$y' = \frac{l \sin \alpha}{4 + \pi \sin \alpha} \left[\frac{\sin \alpha \sin \theta'' (1 - \cos \alpha \cos \theta'')}{(\cos \theta'' - \cos \alpha)^2} - \ln \frac{1 - \cos (\alpha + \theta'')}{\cos \theta'' - \cos \alpha} \right]. \qquad [114k]$$

$$\left(\text{Note that } \frac{1 + \cos (\theta + \alpha)}{\cos \theta + \cos \alpha} = \frac{\cos \theta + \cos \alpha}{1 + \cos (\theta - \alpha)} . \right)$$

The free streamlines for $\alpha = 55°$ are plotted to scale in Figure 186.

The force F_1 on the lamina per unit of its length can be found, as in Equations [113j, k], using $dx = dz$ from Equation [114b], c from Equation [114c], and evaluating $\int_{-\infty}^{-1}$ and $\int_{1}^{\infty}$ separately. When the velocity at infinity and on the free streamlines is U instead of unity,

$$F_1 = \frac{\pi \rho l U^2 \sin \alpha}{4 + \pi \sin \alpha} . \qquad [114l]$$

The force acts perpendicularly on the lamina.

The center of pressure, at which F_1 may be supposed to act, is displaced from the center to the position $x = \bar{x}$ where

$$\bar{x} = \frac{3}{4} \frac{l \cos \alpha}{4 + \pi \sin \alpha} . \qquad [114m]$$

The deduction of $\bar{x}$ is given in Section 77 of Lamb's Hydrodynamics,[1] where Equation [15] is equivalent to Equation [114b] here; a table of values of F_1, x_c and $\bar{x}$ for various values of α is also given.

(For notation and method: see Section 34; Reference 1, Article 77; Reference 2, Section 12.3).

115. INFINITE STREAM ON A V-SHAPED LAMINA.

Let a steady stream impinge symmetrically on a lamina whose section consists of two equal straight lines meeting at an angle 2α as measured on the down-stream side.

The solution in Section 113 can be modified to suit this problem by interposing a transformation

$$\zeta = e^{-i(\frac{1}{2}\pi - \alpha)} \zeta_1^{2\alpha/\pi}. \qquad [115a]$$

Since only values of ζ and ζ_1 on or below the real axis are involved, let all amplitudes θ be taken in the range $-\pi \leq \theta \leq \pi$. Then the positive real axis of ζ_1, on which $\theta = 0$, becomes on the ζ-plane a radius inclined downward at the angle $-\left(\frac{1}{2}\pi - \alpha\right)$; whereas the negative half, on which $\theta = -\pi$, makes with the real axis of ζ the angle $-\left(\frac{1}{2}\pi - \alpha\right) - \left(\frac{2\alpha}{\pi}\right)\pi = -\left(\frac{1}{2}\pi + \alpha\right)$. These two lines on the ζ-plane thus enclose an angle 2α between and below them; and the arc included between them on the unit circle for ζ corresponds to the entire lower half of the unit circle on the ζ_1-plane; see Figure 186, in which the free streamlines are copied from Tumlirz.[188]

If ζ is then assumed to be related to z by the usual Equation [110a] or $dz = -\zeta dw$, the two lines and the enclosed arc correctly represent the dividing streamline, $I''CAI$, $I''CA'I'$, on the z-plane, of which the portions AI and $A'I'$ are free; see Figure 188. The variable ζ_1 may be assumed to be related to t and w in the same way as ζ was in Section 113. Then, from Equation [113c] and Equation [113a],

$$dz = -\zeta dw, \; w = -\frac{c}{t^2}, \; \zeta_1 = -t - (t^2 - 1)^{1/2}. \qquad [115b,c,d]$$

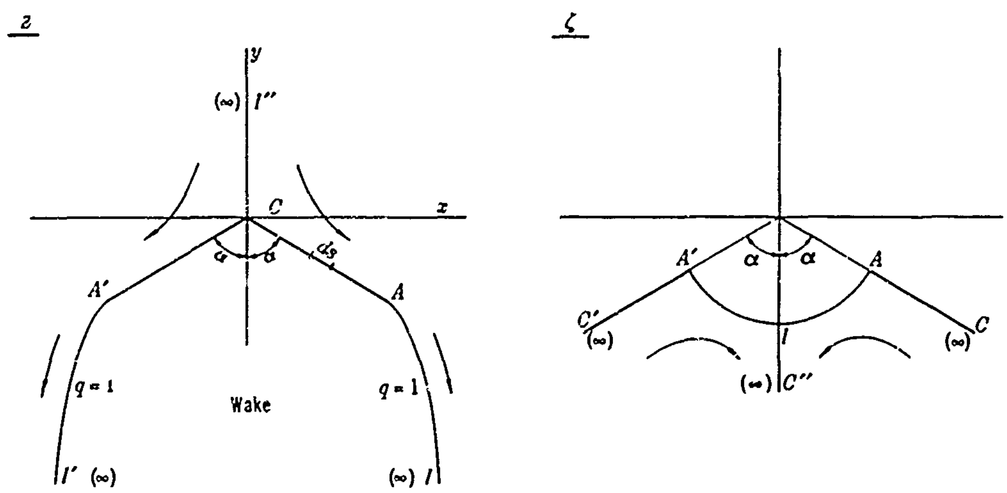


Figure 188 – Symmetrical angle-lamina in a stream with wake behind it. (The free streamlines are copied from Reference 188, Volume 121.)

On the right-hand half of the lamina, t is real and $t \leqq -1$ as before; hence, for continuity, $(t^2 - 1)^{1/2} = -\sqrt{t^2 - 1}$, and

$$\zeta_1 = -t + \sqrt{t^2 - 1}, \quad \frac{1}{\zeta_1} = -t - \sqrt{t^2 - 1}.$$

If ds is an element of length along the lamina and dz the corresponding element of z, from Figure 188

$$dz = dx + i\,dy = ds\,(\sin\alpha - i\cos\alpha) - ds\; e^{-i\left(\frac{1}{2}\pi - \alpha\right)},$$

whence

$$ds = e^{i\left(\frac{1}{2}\pi - \alpha\right)}\,dz = -2c\,\zeta_1^{2\alpha/\pi}\,\frac{dt}{t^3},$$

from Equations [115a, b, c]. Hence, if b is the half-width of the lamina,

$$b = \int ds = -2c\int_{-\infty}^{-1}\left(-t + \sqrt{t^2 - 1}\right)^{2\alpha/\pi}\frac{dt}{t^3}.$$

The integral can be evaluated numerically, and the equation then fixes c in terms of b.

Also, on the right-hand half of the lamina

$$q^2 = \left|\frac{1}{\zeta^2}\right| = \zeta_1^{-4\alpha/\pi},$$

since ζ_1 is real and positive. Hence, the force on the entire lamina per unit of its length, in the direction of the stream, is

$$F_1 = \frac{1}{2}\,\rho\,(\sin\alpha)\int(1 - q^2)\,ds = -2\,c\,\rho\,(\sin\alpha)\int_{-\infty}^{-1}\left(\zeta_1^{2\alpha/\pi} - \zeta_1^{-2\alpha/\pi}\right)\frac{dt}{t^3}$$

$$= -2\,c\,\rho(\sin\alpha)\int_{-\infty}^{-1}\left[\left(-t + \sqrt{t^2-1}\right)^{2\alpha/\pi} - \left(-t - \sqrt{t^2-1}\right)^{2\alpha/\pi}\right]\frac{dt}{t^3}. \qquad [115e]$$

The evaluation of the integrals is discussed in Article 78 of Lamb's Hydrodynamics;[1] the last integral there written is evaluated by complex integration on page 363 of Wilson's Advanced Calculus.[182] In the location first cited is given a table of values of the force, there called "pressure."

(For nonsymmetrical cases; see Reference 2, Section 12.50 and Morton;[197] see Reference 1, Article 78; Reference 2, Section 12.52.)

116. JET ON A WALL.

Let a two-dimensional jet of width a fall upon an infinite plane wall, approaching with uniform velocity U at an angle α with the face of the wall, as in Figure 189. Part of the fluid in the jet will flow away toward one side, part toward the other.

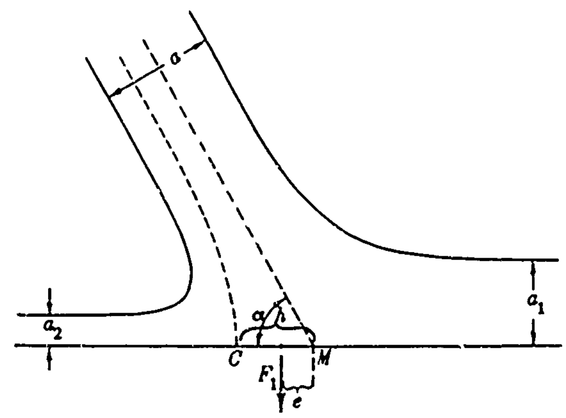


Figure 189 – Two-dimensional jet striking a wall.

In the two departing jets, of widths a_1 and a_2, the velocity will ultimately become uniform and equal to that in the incident jet, since, as in Section 41, it is uniform along the free streamlines. Hence, the incompressibility of the fluid requires that $a_1 + a_2 = a$. Furthermore, the component of momentum parallel to the wall must be conserved, since no force acts on the fluid in this direction. In unit time a mass $\rho a U$ of fluid, carrying momentum $\rho a U^2$, is lost from the incident jet and reappears as masses $\rho a_1 U$ and $\rho a_2 U$ in the departing jets; hence $\rho a U^2 \cos\alpha = \rho a_1 U^2 - \rho a_2 U^2$, and $a \cos\alpha = a_1 - a_2$. From these two equations

$$a_1 = \frac{a}{2}(1 + \cos\alpha), \; a_2 = \frac{a}{2}(1 - \cos\alpha), \; \frac{a_2}{a_1} = \tan^2\frac{\alpha}{2}. \qquad [116a,b,c]$$

The decrease in the component of momentum perpendicular to the wall, on the other hand, equals the total force on the wall, so that

$$F_1 = \rho a U^2 \sin\alpha \qquad [116d]$$

where F_1 is the force per unit of length of the wall in a direction perpendicular to the planes of flow. The effective line of action of F_1 can be found from the conservation of moment of momentum. About the axis M along which the median plane of the incident stream cuts the wall, the incident stream has a zero moment of momentum because of symmetry; but the median planes of the departing jets lie at distances $a_1/2$ and $a_2/2$ from M. Hence, in unit time the combined moment of momentum of these jets increases by

$$\rho a_1 U^2 (a_1/2) - \rho a_2 U^2 (a_2/2) = (a_1^2 - a_2^2)\, \rho U^2/2,$$

in a clockwise direction. The reaction on the wall must be an equal counterclockwise moment of force about M. Hence the force F_1 must act at a point displaced from M to the left, or toward the stagnation point C, through a distance e such that $e F_1 = (a_1^2 - a_2^2)\rho U^2/2$ or, from Equation [116d] and Equations [116a,b],

$$e = \frac{a_1^2 - a_2^2}{2a \sin\alpha} = \frac{a}{2} \cot\alpha . \qquad [116e]$$

Further details can be discovered by resorting to the method of complex variables. Only a few results will be cited here.

The distance h from M to the stagnation point C is

$$h = \frac{a}{2} \cot\alpha + \frac{a}{\pi}\left[(\cos\alpha)\ln(2\sin\alpha) + \ln\cot\frac{\alpha}{2} + \left(\frac{\pi}{2} - \alpha\right)\sin\alpha\right] . \qquad [116f]$$

At *perpendicular incidence* or $\alpha = \pi/2$, the equations for the free streamlines, with the origin taken at the stagnation point C or M and with the x-axis drawn along the wall, are

$$x = \pm a\left(\frac{1}{2} + \frac{1}{\pi}\ln\cot\frac{\theta}{2}\right),\quad y = a\left[\frac{1}{2} + \frac{1}{\pi}\ln\cot\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right], \qquad [116g,h]$$

where θ is the angle, taken positive, between the direction of the tangent to the streamline and the wall. On the median plane, where $x = 0$, in terms of the velocity q, if $U > 0$,

$$y = \frac{a}{\pi}\left[\ln\frac{U+q}{U-q} - 2\tan^{-1}\frac{U}{q} + \pi\right] . \qquad [116i]$$

Along the wall the absolute value of x is given in terms of q by the same expression that represents y along the median plane. Each half of the jet has a plane of symmetry through C inclined at 45 deg to the wall.

Flow nets for $\alpha = \pi/2$ and $\alpha = 3\pi/4$ are reproduced from Reference 183 in Figures 190 and 191. In the figures v_1 stands for U, and ϕ and ψ are reversed in sign in accord with the older convention. The numerical values refer to the case $a = 1$, $v_1 = U = 1$. The broken curves are curves of constant velocity, the value of the ratio $w = q/U$ being indicated for each.

(See Reference 2, Section 11.41; Reference 50 and 183.)

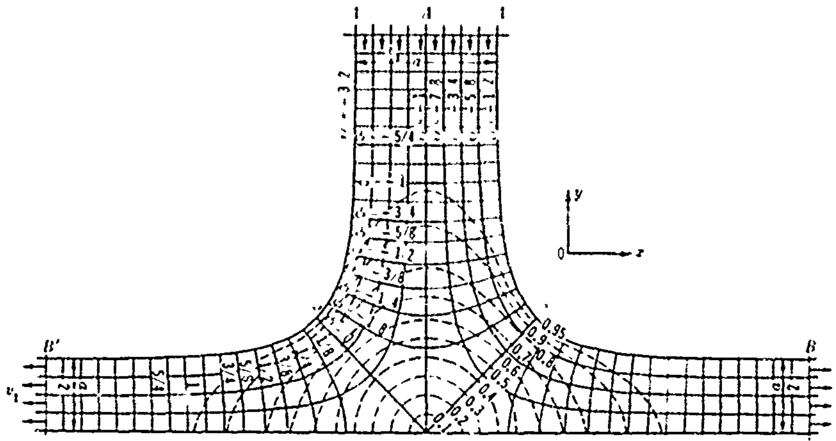


Figure 190 – Flow net for a two-dimensional jet striking a wall perpendicularly (Copied from Reference 183.)

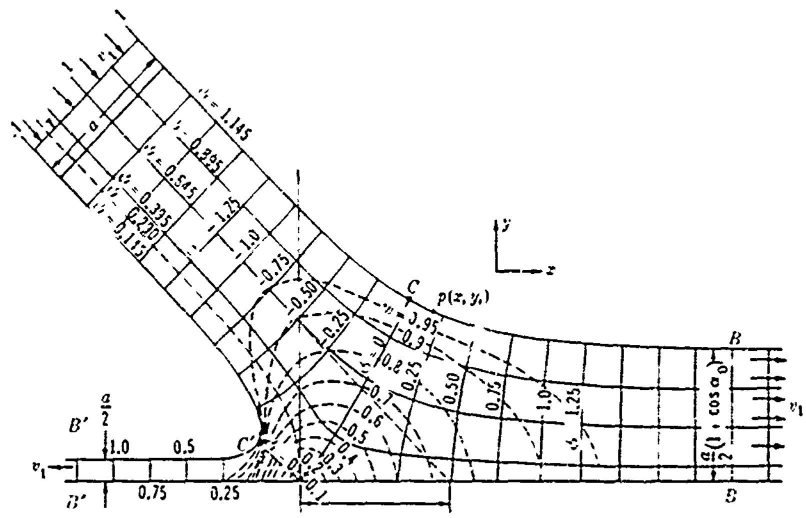


Figure 191 – Flow net for a two-dimensional jet striking a wall at 45 degrees. (Copied from Reference 183.)

117. OTHER FREE-STREAMLINE PROBLEMS.

The literature on potential flow with free streamlines is extensive. A good summary up to 1920 was given by Jaffé.[185] Many authors have followed Levi Civita[186] in using, instead of t, a variable on whose plane the z-plane corresponds to a semicircle; in terms of this variable w is everywhere analytic. The later general discussions given by Cisotti[6] and by Bergmann[187] may be mentioned; see also Reference 2, Sections 12.40–12.47.

Efflux

The efflux of a liquid through a slot in the side of a vessel whose contour is a polygon is easily handled, and many cases have been solved. A simple case is shown in Figure 192a; streamlines inside the vessel and a few equipotential curves are shown for the same case but with a narrower slot in Figure 192b. Three other cases are shown in Figure 193. See Tumlirz,[188] Cisotti,[189] von Mises,[190] and Eck.[191]

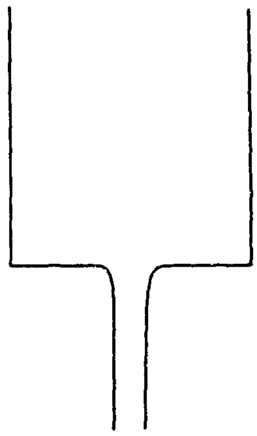


Figure 192a

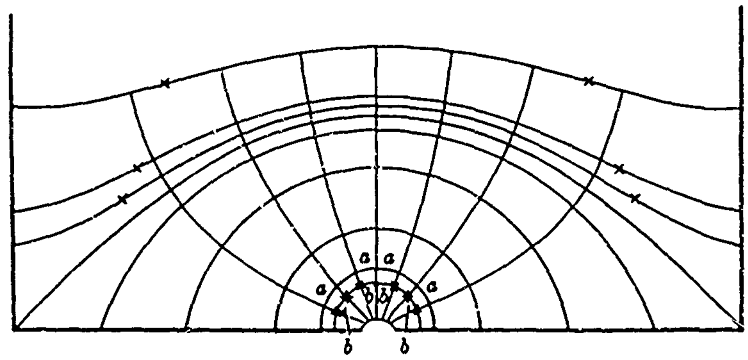


Figure 192b

Figure 192 – Efflux from a tank of finite width. See Section 117.
(Copied from Reference 188, Volume 126.)

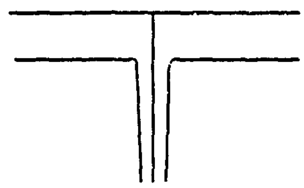


Figure 193a

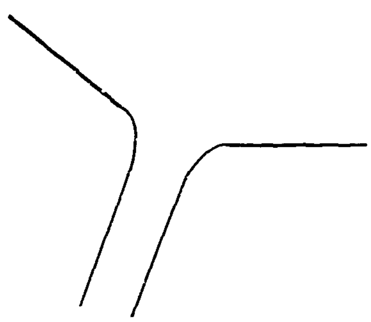


Figure 193b

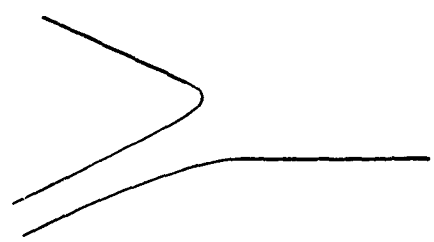


Figure 193c

Figure 193 – Three other cases of efflux; the sides of the issuing jet are shown.
(Copied from Reference 188, Volume 121.)

Efflux from a two-dimensional vase with curved sides was treated by Cisotti;[192] see also Reference 2, Section 11.54.

Deadwater or Wake in an Infinite Stream

A "deadwater," that is a region occupied by stationary fluid or by gas at suitable pressure, ahead of a concave angle was shown to be possible by Villat.[193, 194] A similar deadwater can occur ahead of a convex angle, in addition to the usual wake behind it, when a stream is incident unsymmetrically. The extent of the deadwater is indeterminate; see Thiry,[195] Jaffé,[185, 196] and Morton;[197] see also Yokota.[198] The indeterminateness is perhaps no more surprising than the arbitrariness in the direction of the incident stream; it may be supposed that the size of the deadwater was fixed by the manner in which the flow was established in the first place.

The wake behind a lamina with a rim on the forward side was treated by Love,[50] behind a lamina with flaps folded back by Schmieden,[199] behind a curved lamina by Leathem,[74] Cisotti,[250] and Argeanicoff.[103]

Deadwater regions on the sides of a rectangle immersed in a stream parallel to the sides were described by Riabouchinsky,[110] who also gives values of the inertia coefficient.

A deadwater extending from one plane lamina to another was also described by Riabouchinsky.[110] When the laminas are oppositely inclined to the stream there is circulation around them as a whole.

The wake behind circular and elliptic cylinders placed in a uniform stream was studied by Brodetsky[200] and by Ford,[201] and for the circular case in further detail by Schmieden.[199, 202]

The symmetrically disposed free streamlines behind a *circular cylinder*, on which the velocity is the same as that in the incident stream, may separate from the cylinder at any angular position from $\theta = 55°$ to $\theta = 120°$, approximately, where θ is measured from the stagnation line on the forward side. If they separate at 55 deg, they are concave toward the wake throughout their course; at intermediate angles they are convex near the cylinder and concave beyond; at the largest angle they are convex throughout and meet asymptotically at infinity; see Figures 194, 195.

The flow through a grating of laminas or other cylinders, with a wake behind each, has been considered by von Mises,[190] Betz and Petersohn,[181] and Schmieden.[203]

Free Streamlines in a Channel

The wake behind a body in a channel was considered in a simple case by Cisotti[189] and more generally by Villat[204, 194] and by Bergmann.[187] A deadwater ahead of such a body, or in front of the projecting bend of wall where a channel divides, was considered by Agostinelli.[205] For a channel interrupted by openings where free streamlines occur, see Colonetti,[206] and Miyadzu.[176] The flow past a triangular ridge on a wall with a wake behind the ridge is illustrated in Figure 196, as found by Tumlirz.[188]

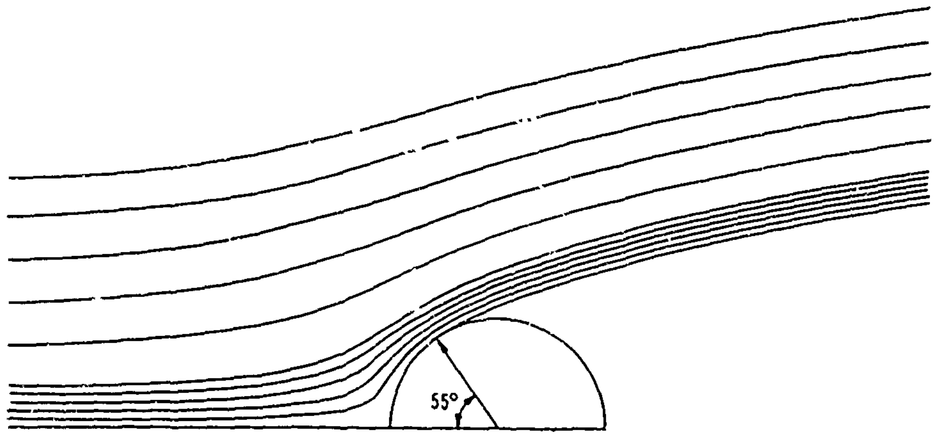


Figure 194 – Some streamlines, on one side of the plane of symmetry, past a cylinder with a wake of maximum size behind it.

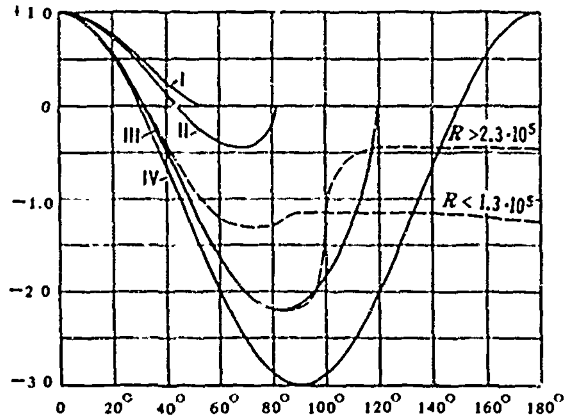


Figure 195 – Pressure on a cylinder in a stream with a wake behind it.

The ordinate represents excess of pressure above pressure at infinity, divided by $\rho U^2/2$ where U is the velocity in the approaching stream. Curves I and III are for the limiting forms of the free streamlines, leaving the cylinder at 55° or 120° ±; these and Curve II continue through the wake along the 0 axis. Curve IV is for the theoretical continuous flow of Section 67. The broken curves represent observed pressures, under conditions of laminar flow ($R < 1.3 \times 10^5$) and of fully turbulent flow ($R > 2.3 \times 10^5$); R is Reynold's number. See Section 117, "Circular Cylinder." (Copied from Reference 199.)

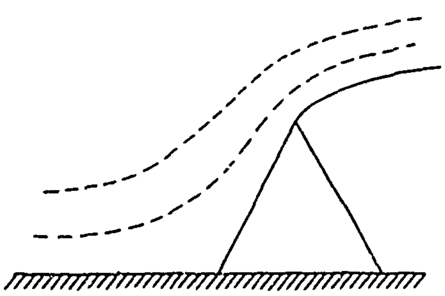


Figure 196 – Streamlines past a wall carrying a triangular bridge, back of which a wake exists. (Copied from Reference 188.)

Free Surface or a Wall

The following cases may be listed: rise of the free surface over a laminar riffle, Cisotti;[207] change in trend of the wall, Boggio;[208] deadwater in an angle, Agostinelli:[205] source against a wall, Masotti.[59]

Gliding of a plate on a free surface is treated by Green[209] and in Milne-Thomson's book (Reference 2, Section 12.3); of a slightly curved plate, by Franke.[210]

A jet of finite width striking a cylinder of the following shape has been studied: a plate, with a wake behind it, by Morton and Harvey,[211] Tomotika,[212] Cisotti,[213] Havelock,[59] Stokalo,[249] or without a wake, by Tomotika;[212] a circular arc without a wake, or a circular cylinder, by Jacob.[211] The jet may issue from a two-dimensional orifice and hit an obstacle; see Valcovici,[215] Tumlirz,[216] and Hartmann.[217] A cylinder may have both a wake behind it and a deadwater in front; see Agostinelli.[205]

Jets or currents of fluid bounded by free surfaces, or in part by walls, may present the following features: branches, Cisotti,[218] also Reference 2, Sections 11.30–11.43; a point of union, Boggio[219] and Caldonazzo,[220] perhaps with one issuing from a vessel, Agostinelli;[221] enclosed deadwater, Caldonazzo,[222] Cisotti,[213] and Brusoni;[223] an enclosed source, Hopkinson[224] and Masotti;[225] an enclosed vortex, Hopkinson,[224] Imai,[226] and Simmons.[227].

A jet issuing from a slot in a plane wall and entering a similar slot in a parallel wall was described by Riabouchinsky.[110]

CHAPTER IV

CASES OF THREE-DIMENSIONAL FLOW

118. INTRODUCTION

In this chapter the principal cases of three-dimensional potential flow that have been worked out will be described, or at least listed. The mathematical solution of problems is much more difficult in three dimensions than in two, since the method of complex variables is no longer available. The usual procedure is to obtain solutions of the Laplace equation by any means whatever, and then by superposing solutions to work out proper combinations for certain specified boundary conditions. A stream function ψ can be defined only for the case of axial symmetry, as described in Section 16.

For the components of the particle velocity of the fluid in the directions of Cartesian or x, y, z axes the symbols u, v, w will be written without repetition of their definition and, as usual, q will denote the speed, so that

$$q = +(u^2 + v^2 + w^2)^{1/2} \qquad [118a]$$

Confusion with the use of w for the complex potential in the last chapter should not arise, since in two-dimensional motion the third velocity component w is always zero. The components of the velocity are understood to be calculated from the velocity potential ϕ by means of the usual equations

$$u = -\frac{\partial\phi}{\partial x}, \quad v = -\frac{\partial\phi}{\partial y}, \quad w = -\frac{\partial\phi}{\partial z} \qquad [118b,c,d]$$

If polar coordinates r, θ, ω are used, or cylindrical coordinates x, ϖ, ω as described in Section 6, the components of the velocity in the corresponding coordinate directions are, as in Equations [6k, l, m, p, q, r],

$$q_r = -\frac{\partial\phi}{\partial r}, \quad q_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta}, \quad q_\omega = \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\omega} \qquad [118e,f,g]$$

or

$$q_x = -\frac{\partial\phi}{\partial x}, \quad q_{\varpi} = -\frac{\partial\phi}{\partial\varpi}, \quad q_\omega = -\frac{1}{\varpi}\frac{\partial\phi}{\partial\omega} \qquad [118h,i,j]$$

When the motion is steady, the pressure is given by the Bernoulli equation, which may be written as in [34h] or

$$p = \frac{1}{2} \rho \, (U^2 - q^2) + p_\infty \qquad [118k]$$

where U is the uniform velocity of the fluid at infinity and p_∞ is the pressure there; ρ is the density of the fluid. Cases in which a body is moving in steady translation with the fluid at rest at infinity can be reduced as usual to the case of steady motion of the fluid by imparting to everything a velocity equal and opposite to that of the body; the pressure and the forces on the body are not thereby affected.

119. POTENTIAL AND STREAM FUNCTIONS FOR A UNIFORM STREAM, A POINT SOURCE OR A POINT DIPOLE

The velocity potential for a *uniform stream* having velocity U toward negative x can be written

$$\phi = Ux \qquad [119a]$$

for then, by (118b,c,d), $u = -U$, $v = 0$, $w = 0$. The streamlines are straight lines parallel to the x-axis.

For some purposes it is convenient to regard such a stream as having axial symmetry about some chosen line parallel to the velocity, and to define an axisymmetric stream function with respect to this line. Let the positive direction along the line be taken toward positive x, and through any point P or (x, y, z) draw a circle of radius ϖ about the chosen line QQ' as axis, as in Figure 197. Then, across any surface bounded by this circle there flows in unit time a volume $\pi\varpi^2 U$ of fluid. Hence, according to the usual definition as stated in Section 16, the axisymmetric or Stokes stream function at P is $\pi\varpi^2 U/2\pi$ or

$$\psi = \frac{1}{2} U\varpi^2 \qquad [119b]$$

In general, any line parallel to the flow may be chosen as the axis of symmetry, and ϖ represents the distance of P from this line. If the x-axis itself is chosen, $\varpi^2 = y^2 + z^2$.

If the velocity of the uniform stream is U toward a direction whose direction cosines are l, m, and n, the velocity potential becomes

$$\phi = -U(lx + my + nz) \qquad [119c]$$

as is easily verified from [119a] by a rotation of axes.

For the *point source*, the velocity potential ϕ at a point (x,y,z) is, from Section 12,

$$\phi = \frac{A}{r} \qquad [119d]$$

where r denotes distance of (x,y,z) from the location of the source, and A is a constant, positive for an actual source and negative for a sink. The quantity $4\pi A$ represents the volume of fluid emitted by the source per second. The streamlines are radii drawn from the source.

An axis of symmetry may be drawn through the source in any direction; let it be taken as the axis of polar coordinates r, θ, ω with origin at the source. Let a circle be drawn as before, with the axis of symmetry as its axis, but now consider this circle as the perimeter of a spherical cap C cut out of a sphere centered at the source; see Figure 197 again, where a source is now understood to be located at Q, and $U = 0$. The area of this cap is $S = 2\pi r^2 (1 - \cos\theta)$, where r and θ refer to any point on the circle, and the rate of outflow of fluid across it is $S_q = SA/r^2$, because of the symmetry. Hence the value of the stream function at P is $-SA/r^2$ divided by 2π or, after inserting the value of S and dropping the constant term $-A$,

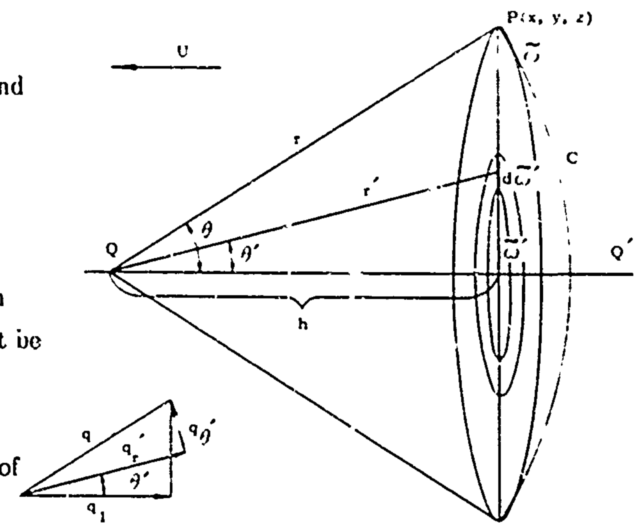


Figure 197 – Symbols for flow symmetric about a line $Q\,Q'$.

$$\psi = A \cos\theta \qquad [119e]$$

The total range in the values of ψ from $\theta = 0$ to $\theta = \pi$ is thus $-2A$, which equals the volume output from the source per second or $4\pi A$ reversed in sign and divided by 2π.

If Cartesian axes are introduced and the x-axis is drawn parallel to the axis of symmetry and toward $\theta = 0$, and if the source is at (x_1, y_1, z_1) as in Figure 198, then

$$r = \left[(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2\right]^{1/2}$$

and

$$\psi = A\,\frac{x-x_1}{r} \qquad [119f]$$

For a *point dipole*, let polar coordinates r, ϑ, ω be employed for the moment, with the origin at the dipole and the axis for θ lying along the axis of the dipole. Then the potential is, as in [12d]

$$\phi = \frac{\mu\cos\theta}{r^2} \qquad [119g]$$

where μ is a constant, the dipole moment. The physical axis of the dipole is directed toward $\theta = 0$ if $\mu > 0$, toward $\theta = \pi$ if $\mu < 0$.

The fluid velocity lies in a plane through the axis of the dipole, which is an axis of symmetry. Its components in the r, θ directions are, from [118e,f,g],

$$q_r = \frac{2\mu \cos\theta}{r^3}, \quad q_\theta = \frac{\mu \sin\theta}{r^3}, \quad q_\omega = 0 \qquad [119h,i,j]$$

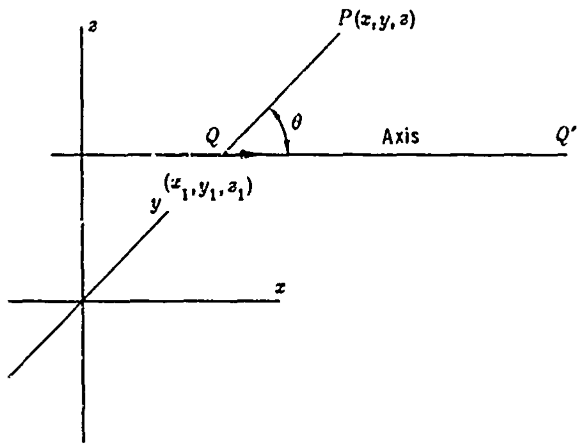


Figure 198 – A dipole at Q or (x_1, y_1, z_1), with its axis parallel to the x-axis.

The stream function ψ, like ϕ, is most easily found by differentiating that for a point source, but it can also be found by direct integration. Through any point P draw a circle as before, with the axis of the dipole as its axis; let its radius be $\tilde{\omega}$, and let the distance of its plane from the dipole at Q be h. On the plane of the circle take a ring-shaped element of area centered on the axis; see Figure 197 again, in which there is now a dipole at Q with its axis along QQ', and again $U = 0$. The area of the ring will be $2\pi\tilde{\omega}' d\tilde{\omega}'$ where $d\tilde{\omega}'$ is its width and $\tilde{\omega}'$ is the radius of either perimeter; and, because of the symmetry, the rate of flow across it will be $2\pi q_1 \tilde{\omega}' d\tilde{\omega}'$ where q_1 is the component of the velocity in the direction of the axis. The flow across the entire circle is thus

$$\int_0^{\tilde{\omega}} q_1 2\pi\tilde{\omega}' d\tilde{\omega}'$$

Now, by projection, if r', θ' are the values of r and θ on the elementary ring,

$$q_1 = q_r \cos\theta' - q_\theta \sin\theta' = \frac{\mu}{r'^3}(2\cos^2\theta' - \sin^2\theta')$$

$$= \frac{\mu}{r'^3}(3\cos^2\theta' - 1) = \mu\left(\frac{3h^2}{r'^5} - \frac{1}{r'^3}\right)$$

See Figure 197. Thus the flow across the circle is

$$2\pi\mu \int_0^{\widetilde{\omega}} \left(\frac{3h^2}{r'^5} - \frac{1}{r'^3} \right) \widetilde{\omega}' d\widetilde{\omega}' = 2\pi\mu \frac{\widetilde{\omega}^2}{r^3}$$

since $r' = (h^2 + \widetilde{\omega}^2)^{1/2}$. Hence, if the positive axis for the dipole moment is chosen also as the positive axis in defining the axisymmetric stream function ψ,

$$\psi = -\mu \frac{\widetilde{\omega}^2}{r^3} \qquad [119k]$$

Here $\widetilde{\omega}$ denotes distance from the axis of the dipole. If x denotes distance along the axis, measured from the dipole,

$$r = [x^2 + \widetilde{\omega}^2]^{1/2}, \ \cos\theta = \frac{x}{r}, \ \sin\theta = \frac{\widetilde{\omega}}{r}$$

The variables x and $\widetilde{\omega}$ may also be regarded as two out of a set of cylindrical coordinates $x, \widetilde{\omega}, \omega$, and in terms of them the potential and the corresponding components of velocity are

$$\phi = \frac{\mu x}{r^3}; \quad q_x = \frac{\mu}{r^3}\left(3\frac{x^2}{r^2} - 1\right), \quad q_{\widetilde{\omega}} = 3\mu \frac{x\widetilde{\omega}}{r^5} \qquad [119l,m,n]$$

and $q_\omega = 0$.

Or, if Cartesian coordinates are used, with the x-axis drawn in the direction of the dipole moment for positive μ, and if the dipole is at (x_1, y_1, z_1), as in Figure 198,

$$r = [(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2]^{1/2}, \quad \widetilde{\omega} = [(y-y_1)^2 + (z-z_1)^2]^{1/2}$$

and the potential and the three velocity components may be written, from [119g] and [118b,c,d],

$$\phi = \mu \frac{x - x_1}{r^3} \qquad [119o]$$

$$u = \frac{\mu}{r^3}\left(\frac{3(x-x_1)^2}{r^2} - 1\right), \quad v = \mu \frac{3(x-x_1)(y-y_1)}{r^5}, \quad w = \mu \frac{3(x-x_1)(z-z_1)}{r^5}$$

[119p,q,r]

If the x-axis is otherwise drawn, let the direction of the dipole axis for positive μ have direction cosines l, m, n. Then, by rotation of axes it is seen that

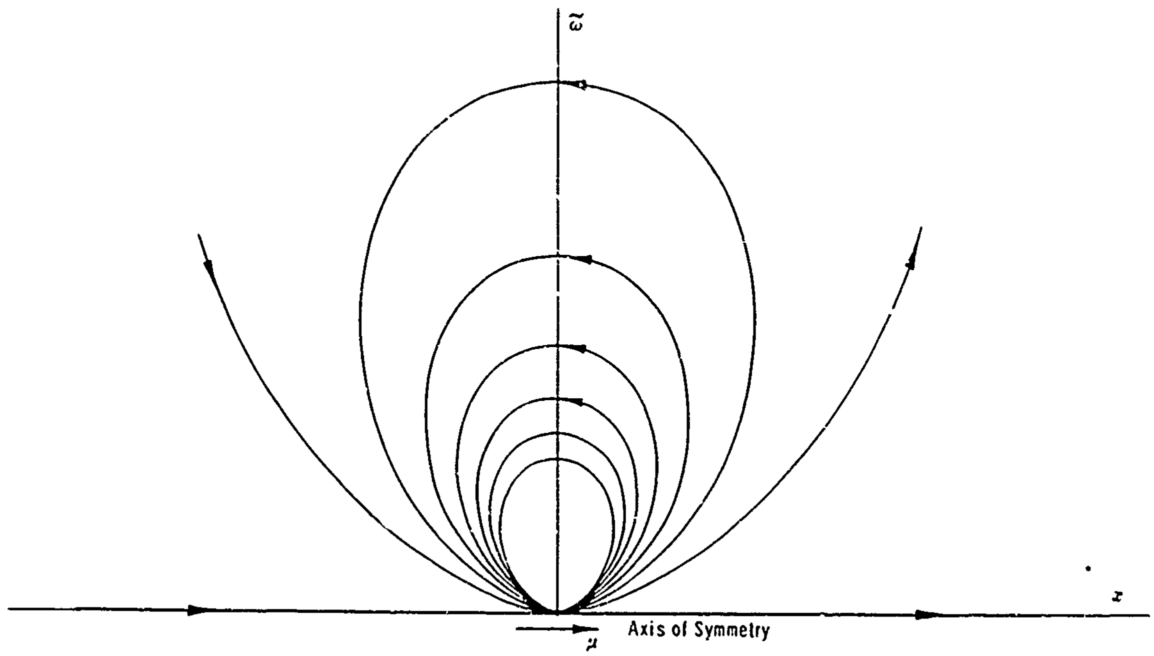


Figure 199 – Some streamlines due to a dipole, for equally spaced values of ψ.

$$\phi = \frac{\mu}{r^3} [l(x - x_1) + m(y - y_1) + n(z - z_1)] \qquad [119s]$$

Since $m\mu(y - y_1)/r^3$ and $n\mu(z - z_1)/r^3$ obviously represent potentials due to dipoles with axes directed toward y or toward z, respectively, the potential can be regarded as the sum of the potentials due to three component dipoles having moments $l\mu$, $m\mu$, $n\mu$; these moments represent the vector components of the moment μ regarded as a vector.

Streamlines selected for equally spaced values of ψ, including the x-axis, are shown in Figure 199, which may refer to any plane through the dipole axis. Only half of the symmetric diagram in this plane is shown.

Reversing the sign of μ reverses the direction of the axis of the dipole and reverses all velocities. As an alternative, μ may be kept positive and the axis for θ may be drawn in the opposite direction. (See Reference 1, Article 95; Reference 2, Section 15.20, 15.22, 15.26.)

120. VARIABLE POINT SOURCE, OR FLOW NEAR A SPHERICAL CAVITY

In the flow from a point source, the potential ϕ and the velocity q_r, taken positive when directed outward from the source, are

$$\phi = \frac{A}{r}, \quad q_r = \frac{A}{r^2} \qquad [120a,b]$$

where r denotes distance from the source. The pressure at any point in the fluid, from the usual pressure equation for incompressible frictionless fluid moving irrotationally or Equation [9g], is

$$p = \rho \left(\frac{\partial \phi}{\partial t} - \frac{1}{2} q^2\right) + \text{constant} = \rho \left(\frac{1}{r}\frac{dA}{dt} - \frac{A^2}{2r^4}\right) + p_\infty \qquad [120c]$$

where p_∞ is the pressure at infinity. For, at a given point, only the factor A in ϕ varies, so that

$$\frac{\partial \phi}{\partial t} = \frac{1}{r}\frac{dA}{dt}$$

The point source is an ideal abstraction that is useful in building up solutions of practical problems.

Consider, for example, a sphere whose radius R varies with the time; or, it may be simply a spherical cavity in the fluid, or a bubble of gas. Let the fluid motion be spherically symmetrical about the center P of the sphere or cavity. Then it can be represented by the formulas appropriate to a point source located at P; see Figure 200.

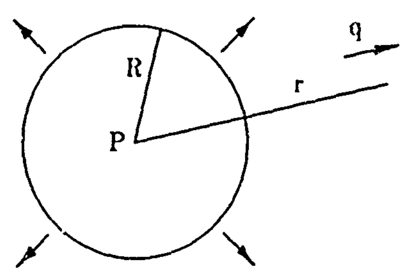


Figure 200 – A spherical cavity of variable radius R.

At the sphere, $r = R$ and $q_r = dR/dt$. Thus, from [120b],

$$A = R^2 \frac{dR}{dt} \qquad [120d]$$

and in the surrounding fluid

$$\phi = \frac{R^2}{r}\frac{dR}{dt}, \quad q_r = \left(\frac{R}{r}\right)^2 \frac{dR}{dt} \qquad [120e,f]$$

$$p = \rho \left[\frac{1}{r}\frac{d}{dt}\left(R^2 \frac{dR}{dt}\right) - \frac{1}{2}\left(\frac{R}{r}\right)^4 \left(\frac{dR}{dt}\right)^2\right] + p_\infty \qquad [120g]$$

If the pressure in the cavity or bubble is p_R, then $p_R = p$ at $r = R$ or

$$p_R = \rho\left[\frac{1}{R}\,\frac{d}{dt}\left(R^2\,\frac{dR}{dt}\right) - \frac{1}{2}\left(\frac{dR}{dt}\right)^2\right] + p_\infty \qquad [120h]$$

If p_R is known as a function of R, this differential equation determines R as a function of the time.

The kinetic energy of the fluid is

$$T = \frac{1}{2}\,\rho \int q_r^{\,2}\,(4\pi r^2)\,dr = 2\pi\rho A^2 \int_R^\infty \frac{dr}{r^2} = 2\pi\rho\,\frac{A^2}{R} = 2\pi\rho R^3\left(\frac{dR}{dt}\right)^2 \qquad [120i]$$

from [120d]. (See Reference 1, Articles 56, 91a; Reference 2, Section 15.20.)

121. POINT SOURCE IN A UNIFORM STREAM

Let the flow due to a point source be superposed upon a uniform streaming motion. Take the origin 0 at the source and the x-axis parallel to the flow at infinity. Since the motion is then axisymmetric about the x-axis, it suffices to take as a second coordinate the distance ϖ from the axis, and to study the flow in a single plane; see Figure 201.

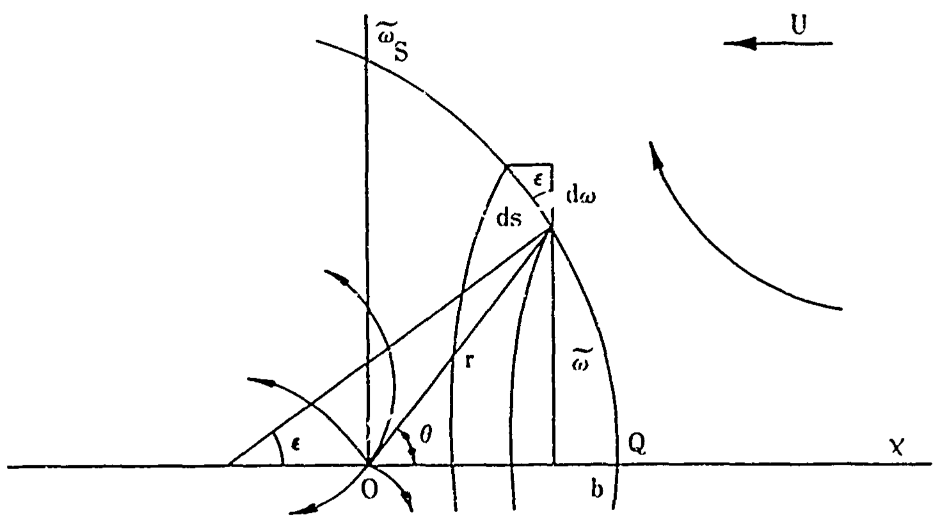


Figure 201 - A point source at 0, in a uniform stream. See Section 121.

The potential and stream functions can be written, from Equations [119a,b] and [119f],

$$\phi = U\left(x + \frac{b^2}{r}\right), \quad \psi = U\left(\frac{1}{2}\,\varpi^2 + b^2\,\frac{x}{r}\right), \quad r = \sqrt{x^2 + \varpi^2} \qquad [121a,b,c]$$

Here b is a positive constant and U denotes the velocity at infinity, taken positive when directed toward negative x. The x and ϖ components of velocity are, from [118h,i],

$$q_x = -\frac{\partial\phi}{\partial r} = -U\left(1 - b^2\,\frac{x}{r^3}\right), \quad q_{\widetilde{\omega}} = -\frac{\partial\phi}{\partial\widetilde{\omega}} = b^2\,U\,\frac{\widetilde{\omega}}{r^3} \qquad [121\text{d,e}]$$

whence

$$q^2 = q_x^{\,2} + q_{\widetilde{\omega}}^{\,2} = U^2\left(1 - 2\,b^2\,\frac{x}{r^3} + \frac{b^4}{r^4}\right) \qquad [121\text{f}]$$

A stagnation point Q occurs on the [illegible] make $q = 0$, $\widetilde{\omega} = 0$, so that $x = r = b$.

Clearly on the entire positive x-axis $\psi = b^2U$ and is constant. But $\psi = b^2U$ also on a surface of revolution S defined by the equation

$$\widetilde{\omega}^2 = 2b^2\left(1 - \frac{x}{r}\right) = 2b^2\,(1 - \cos\theta) \qquad [121\text{g}]$$

where $\cos\theta = x/r$. On this surface $\widetilde{\omega}$ has a maximum value of $2b$ at $\theta = \pi$, or as $x \to -\infty$ and $r \to |x|$. As x increases algebraically, $\widetilde{\omega}$ decreases, and vanishes when $\theta = 0$. To find x at this latter point, substitute $r = \sqrt{x^2 + \widetilde{\omega}^2}$ and rationalize, obtaining

$$\widetilde{\omega}^4 + (x^2 - 4b^2)\,\widetilde{\omega}^2 + 4b^2\,(b^2 - x^2) = 0 \qquad [121\text{h}]$$

At $\widetilde{\omega} = 0$, $x = b$. Also, differentiating [121h], $[4\widetilde{\omega}^3 + 2(x^2 - 4b^2)\widetilde{\omega}]\,(d\widetilde{\omega}/dx) + 2(\widetilde{\omega}^2 - 4b^2)\,x = 0$, whence, as $\widetilde{\omega} \to 0$, $d\widetilde{\omega}/dx \to -\infty$. Hence the surface S cuts the x-axis perpendicularly at the stagnation point Q or $x = b$.

Thus the streamline for $\psi = b^2U$, approaching from both sides along the x-axis, divides at Q into a sheaf of lines that extend off to infinity along S. The surface S divides space into an exterior region occupied by fluid belonging to the incident stream and an interior region occupied by fluid that has come from the source.

If a solid boundary is introduced along S, no singularities occur outside it. Hence the formulas represent flow past a body of this shape, or a blunt-nosed cylinder of asymptotic diameter $4b$. Its shape is fixed uniquely, since, if b is changed, Equation [121g] remains satisfied when all coordinates are changed in proportion to b.

In Figure 202 are shown some of the streamlines on a typical plane through the axis of symmetry; the lines are equally spaced at infinity and thus differ by equal increments of the quantity $\psi/\widetilde{\omega}$. The excess of the pressure p above the pressure p_∞ in the stream at infinity, when the motion is steady, is also plotted, for points on S or on the x-axis in front of it. On S, $p = p_\infty$ at $2r\,x = b^2$ or $x = b/\sqrt{6} = 0.408b$.

To find the total force on the solid, which must be parallel to the axis by symmetry, select a narrow ring cut from its surface by two planes perpendicular to the axis, as illustrated in Figure 200. The circumference of the ring is $2\pi\widetilde{\omega}$, hence its area is $2\pi\widetilde{\omega}\,ds$,

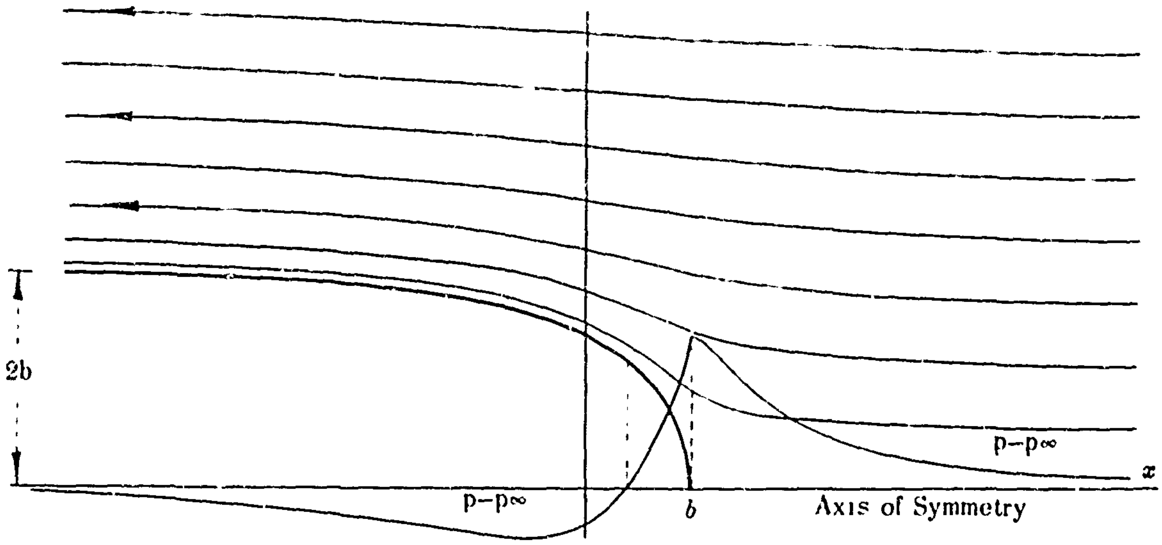


Figure 202 – Streamlines past a semi-infinite solid of revolution obtained from a point source at 0, and plot of the pressure along the axis and over the solid. See Section 121

where ds is its width along the tangent to S in a plane through the x-axis. Let the normal to the surface at any point on the ring meet the axis at an angle ϵ . Then the force due to the pressure p on any element of the ring has a component along the axis equal to the force multiplied by cos ϵ , and, since p and ϵ are uniform around the ring, the total component of force due to the ring is $dF = p\,(2\pi\widetilde{\omega}\,ds)\cos\epsilon$. But $d\widetilde{\omega} = ds \cos\epsilon$ where $d\widetilde{\omega}$ is the element of $\widetilde{\omega}$ corresponding to ds. Hence the total force on the solid, measured positively toward negative x, is

$$F = 2\pi \int p\widetilde{\omega}\,d\widetilde{\omega} \qquad [121i]$$

This formula holds for any surface and any pressure distribution which have a common axis of symmetry.

For steady motion, the excess pressure, $p - p_\infty = \rho\,(U^2 - q^2)/2$, may be inserted for p in this formula. It is simpler to change to r as a variable of integration. Substituting $x^2 = r^2 - \widetilde{\omega}^2$ in Equation [121h], and then differentiating,

$$\widetilde{\omega}^2 = 4b^2 - \frac{4b^4}{r^2}, \quad 2\widetilde{\omega}\,d\widetilde{\omega} = \frac{8b^4}{r^3}\,dr \qquad [121j,k]$$

Eliminating x/r from Equation [121f] by means of [121g] and [121j],

$$q^2 = U^2\left(1 + \frac{2b^2}{r^2} - \frac{3b^4}{r^4}\right) \qquad [121l]$$

on the solid. The limits of integration for r are from $r = b$ at Q to ∞. Evaluation of the integral gives

$$F = 2\pi \int \frac{\rho}{2} \left(U^2 - q^2\right) \varpi \, d\varpi = 0$$

Thus, if the motion is steady, the total force on the solid is the same as if the pressure in the fluid were uniform and equal to its actual value at infinity.

The formulas would also represent the flow inside a shell having the shape of S, due to a source on its axis. The volume of fluid emitted per second by the source is $4\pi b^2 U$.

Changing the sign of U merely reverses the velocity at all points. To reverse the solid lengthwise, the x-axis may be drawn in the opposite direction. (See Reference 2, Section 15.23.)

122. POINT SOURCE AND SINK IN UNIFORM STREAM; RANKINE SOLIDS

Upon a uniform stream with velocity U in the direction of negative x, superpose the flow due to a point source on the x-axis at $x = a$ and also that due to a sink of equal strength at $x = -a$. The resulting potential and stream function, from Equations [119a,b] and [119d,e], can be written

$$\phi = U \left[x + \frac{b^2}{2} \left(\frac{1}{r_1} - \frac{1}{r_2} \right) \right] \qquad [122a]$$

$$\psi = \frac{1}{2} U [\varpi^2 + b^2 (\cos \theta_1 - \cos \theta_2)] \qquad [122b]$$

where b is a positive constant and the significance of r_1, r_2, θ_1, θ_2 is shown in Figure 203. The figure refers to any plane through the x-axis, about which the flow is symmetrical. In particular,

$$r_1 = [(x-a)^2 + \varpi^2]^{1/2}, \quad r_2 = [(x+a)^2 + \varpi^2]^{1/2}$$

The flow net is symmetrical also with respect to the plane $x = 0$. For, the second term in the brackets in [122b] can also be written $-b^2[\cos(\pi - \theta_1) + \cos \theta_2]$, and it is then clear that ψ is unaltered whereas ϕ is reversed in sign if x, r_1, and $\pi - \theta_1$ are interchanged with $-x$, r_2, and θ_2.

The components of velocity are, from [118h,i,j], $q_\omega = 0$ and

$$u = U \left[-1 + \frac{b^2}{2} \left(\frac{x-a}{r_1^3} - \frac{x+a}{r_2^3} \right) \right] \qquad [122c]$$

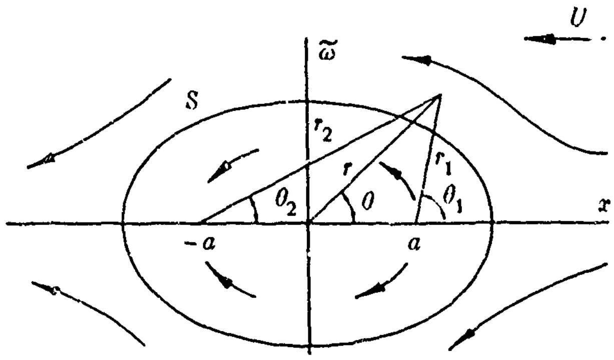


Figure 203 – Point source and sink at $\pm a$, in a stream. See Section 122.

$$q_{\tilde{\omega}} = \frac{1}{2} b^2 U \tilde{\omega} \left(\frac{1}{r_1^3} - \frac{1}{r_2^3} \right) \qquad [122d]$$

On the x-axis wherever $x > a$, $r_1 = x - a$ and $r_2 = x + a$; where $x < -a$, $r_1 = a - x$, $r_2 = -(a + x)$. Hence at such points

$$u = U \left(-1 + \frac{2ab^2 |x|}{(x^2 - a^2)^2} \right) \qquad [122e]$$

and $q = |u|$. Stagnation points Q_1 and Q_2 occur where $x = \pm l$ and l is given by

$$2ab^2 l = (l^2 - a^2)^2 \qquad [122f]$$

In the plane $x = 0$, $q = |u|$, $r_1 = r_2 = \sqrt{\tilde{\omega}^2 + a^2}$, and

$$u = -U \left(1 + \frac{ab^2}{(\tilde{\omega}^2 + a^2)^{3/2}} \right) \qquad [122g]$$

On the x-axis, $\psi = 0$ where $x > a$ or $x < -a$, so that $\theta_1 = \theta_2$. Between $x = \pm a$, $\theta_1 = \pi$, $\theta_2 = 0$, $\psi = -b^2 U$.

The value $\psi = 0$ occurs also on the surface of revolution S defined by the equation

$$\tilde{\omega}^2 = b^2 (\cos \theta_2 - \cos \theta_1) \qquad [122h]$$

By writing $\cos \theta_2 - \cos \theta_1 = (\cos^2 \theta_2 - \cos^2 \theta_1)/(\cos \theta_2 + \cos \theta_1)$ and then expressing $\cos \theta_1$ and $\cos \theta_2$ in terms of x, $\tilde{\omega}$, and a, the equation can also be put into the form

$$4ab^2 x = r_1 r_2 [(x+a) r_1 + (x-a) r_2] \qquad [122i]$$

It is then easily seen that the surface passes through both stagnation points. It is broadest in the middle; its half width h is the value of $\tilde{\omega}$ when $x = 0$ and is given by

$$h^2 = 2\, ab^2/\sqrt{a^2 + h^2} \qquad [122j]$$

since $\cos\theta_2 = -\cos\theta_1 = a/\sqrt{a^2+h^2}$ when $x = 0$.
This equation and [112f] can also be written

$$\left(\frac{h}{a}\right)^2 = 2\left(\frac{b}{a}\right)^2 \left(1 + \frac{h^2}{a^2}\right)^{-1/2}, \quad 2\,\frac{l}{a}\left(\frac{b}{a}\right)^2 = \left(\frac{l^2}{a^2} - 1\right)^2$$

which shows that the shape, being fixed when h/a and l/a are known, depends only on the constant b/a.

The surface S acts again as a dividing surface. The fluid brought up by the stream remains outside of S; the space inside it is occupied by fluid that is on its way from the source to the sink. The streamline $\psi = 0$ follows the x-axis to Q_1, divides into a sheaf of lines which pass around S to reunite at Q_2, and continues along the x-axis.

The formulas may represent streaming flow past a solid whose surface is S. Solids having such shapes are called Rankine solids. Given the length $2l$ and the maximum breadth $2h$ of the solid, a and b can be found from [122f] and [122j]. The velocity is most conveniently found by adding vectorially the component velocities due to the stream and the two sources.

An example of the streamlines is shown in Figure 204, for $b^2/a^2 = 0.7$, $h/a = 0.97$, $l/a = 1.58$. Streamlines are drawn for equally spaced values of $\psi/\tilde{\omega}$.

The formulas could also be used for the flow inside a shell having the shape of S, caused by a source and a sink at the proper points.

If $U > 0$, there is a positive source at $x = a$, and a sink at $x = -a$. If $U < 0$, all velocities are reversed and the source and sink are interchanged, but the solid is unaffected. (See Reference 1, Article 97; Reference 2, Section 15.27.)

123. LINE DISTRIBUTIONS OF POINT SOURCES

For some purposes it is useful to imagine point sources distributed continuously along a line or curve. Let the algebraic strength of the sources per unit of length along the curve be α, so that, from a length ds, $4\pi\ \alpha ds$ units of volume of fluid are emitted per second. Then from [119d], in which αds replaces A, the potential due to the sources on ds at a distance r from ds will be $\alpha ds/r$, and the total potential at any point (x, y, z) due to all sources on the curve will be

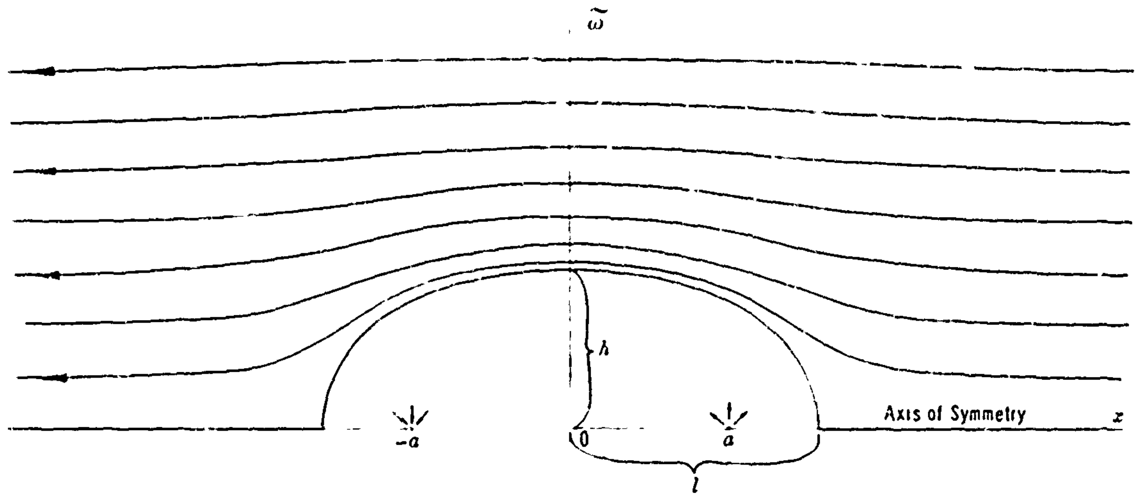


Figure 204 – Streamlines past a rankine solid of revolution, obtained from a source and sink at $\pm a$.

$$\phi = \int \frac{\alpha\, ds}{r} \qquad [123a]$$

where r denotes distance from ds to (x, y, z) and the integral is to be extended over the entire curve. Both α and r may vary along the curve.

The uniform line distribution. As an important special case, let α be uniform along the x-axis from $x = a$ to $x = b$. In the integral for ϕ write dx' for ds and x' for the value of x at ds. Then

$$\phi = \alpha \int_a^b \frac{dx'}{r'}, \quad r' = [(x - x')^2 + \tilde{\omega}^2]^{1/2} \qquad [123b,c]$$

where $\tilde{\omega}$ denotes distance from the x-axis. In this case an antisymmetric stream function ψ also exists; integration of [119f] gives for it

$$\psi = \alpha \int_a^b \frac{x - x'}{r'}\, dx'$$

See Figure 205.

Evaluation of the integrals gives

$$\phi = -\alpha \ln \left\{ [(x' - x)^2 + \tilde{\omega}^2]^{1/2} - (x' - x) \right\} \Big/_{x'=a}^{x'=b}$$

or

$$\phi = \alpha \ln \frac{r_a + x - a}{r_b + x - b} = \alpha \ln \frac{r_a + r_b + b - a}{r_a + r_b - b + a} \qquad [123d]$$

$$\psi = \alpha (r_a - r_b) \qquad [123e]$$

where r_a and r_b denote distances from the ends of the line of sources, or

$$r_a = [(x-a)^2 + \tilde{\omega}^2]^{1/2}, \quad r_b = [(x-b)^2 + \tilde{\omega}^2]^{1/2}$$

The identity of the two forms given for ϕ is easily verified by eliminating x.

The components of velocity are, from [118h,i],

$$q_x = -\frac{\partial \phi}{\partial x} = \alpha \left(\frac{1}{r_b} - \frac{1}{r_a} \right), \quad q_{\tilde{\omega}} = -\frac{\partial \phi}{\partial \tilde{\omega}} = \frac{\alpha}{\tilde{\omega}} \left(\frac{x-a}{r_a} - \frac{x-b}{r_b} \right) \qquad [123f,g]$$

since $\tilde{\omega}^2 = r_a^2 - (x-a)^2 = r_b^2 - (x-b)^2$.

The equipotential or ϕ surfaces are ellipsoids, the stream surfaces for ψ = constant are hyperboloids; all have common foci at $(a,0,0)$, $(b,0,0)$.

If $\alpha < 0$, there is a line of sinks instead of actual sources. (See Reference 2, Section 15.24; Reference 7, p.60.)

124. LINE OF POINT SOURCES IN A STREAM

Suppose that a uniform distribution of point sources exists along the stretch of the x-axis from $x = -a$ to $x = 0$, and that the fluid at infinity is also streaming at velocity U toward negative x; see Figure 205. From [119a,b] and [123d,e], in which now $a \to -a$, $b \to 0$,

$$\phi = Ux + \alpha \ln \frac{r_1 + x + a}{r + x} = Ux + \alpha \ln \frac{r + r_1 + a}{r + r_1 - a} \qquad [124a]$$

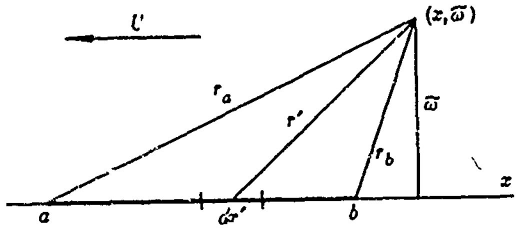
$$\psi = \frac{1}{2} U \tilde{\omega}^2 + \alpha (r_1 - r) \qquad [124b]$$

where

$$r = (x^2 + \tilde{\omega}^2)^{1/2}, \quad r_1 = [(x+a)^2 + \tilde{\omega}^2]^{1/2}$$

and $\tilde{\omega}$ denotes distance from the x-axis. The volume of fluid emitted per second from unit length of the line of sources is $4\pi\alpha$. Let α and U have the same sign.

Figure 205 – A line distribution ab of point sources. See Section 123.



The components of velocity are, using Equation [123f,g] with $a=-a$, $b=0$, $r_a=r_1$, $r_b=r$,

$$q_x = -U + \alpha\left(\frac{1}{r} - \frac{1}{r_1}\right), \quad q_{\varpi} = \frac{\alpha}{\varpi}\left(\frac{x+a}{r_1} - \frac{x}{r}\right) \qquad [124c,d]$$

On the positive x-axis $r = x$, $r_1 = x + a$; and it is obvious from symmetry that $q_{\varpi} = 0$. Hence a stagnation point Q occurs where, to make $q_x = 0$,

$$x = x_Q = \frac{a}{2}\left(\sqrt{1 + 4\frac{\alpha}{aU}} - 1\right) \qquad [124e]$$

On the positive x-axis $\psi = \alpha a$; also, $\psi = \alpha a$ on the surface of revolution S defined by

$$\varpi^2 = \frac{2\alpha}{U}(a + r - r_1) \qquad [124f]$$

By expanding r and r_1 in powers of ϖ^2, it is readily shown that S crosses the x-axis perpendicularly at Q. Since everywhere on S

$$\varpi\frac{d\varpi}{dx} = \frac{\alpha}{U}\left(\frac{x}{r} - \frac{x+a}{r_1}\right) = \frac{\alpha}{U}(\cos\theta - \cos\theta_1) < 0$$

the surface S is broadest at $x = -\infty$, where, since ϖ cannot increase without limit on S, $r - r_1 \to a$ and $\varpi \to 2\sqrt{\alpha a/U}$. For the definition of θ and θ_1 see Figure 206. Thus, if R is the maximum radius of S;

$$R = 2\sqrt{\frac{\alpha a}{U}}, \quad x_Q = \frac{1}{2}\left(\sqrt{a^2 + R^2} - a\right) \qquad [124g,h]$$

At the middle of the line of sources or at $x = -a/2$, where $r = r_1$, on S

$$\varpi = h = \sqrt{\frac{2\alpha a}{U}} = \frac{R}{\sqrt{2}} \qquad [124i]$$

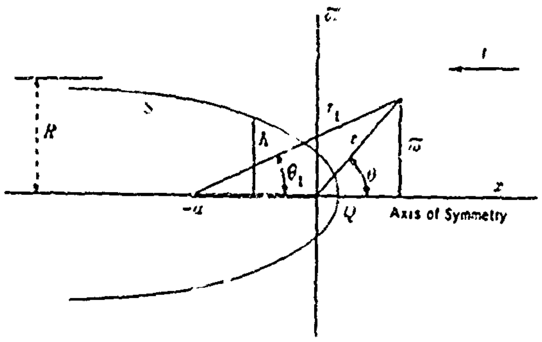


Figure 206 – Dividing surface S for a uniform line of sources in a stream. See Section 124.

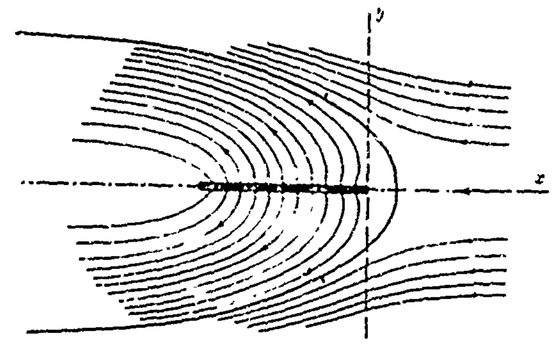


Figure 207 – Streamlines for a uniform line of sources in a uniform stream. The heavy curve is the dividing surface S. See Section 124. (Copied from Reference 7.)

The surface S is a dividing surface, and the formulas may represent the flow past a solid of revolution whose surface is S. Its shape depends upon the dimensionless quantity α/aU, and its size upon the length a of the line of sources, since [124f] can be written

$$\left(\frac{\varpi}{a}\right)^2 = \frac{2\alpha}{aU}\left(1 + \frac{r - r_1}{a}\right)$$

so that, for fixed α/aU, all dimensions vary as a. If $a \to 0$ while αa remains constant, the shape becomes that of Section 121.

Streamlines drawn for equally spaced values of ψ/ϖ, for the same shape of S as in Figure 206, are shown in Figure 207. Here $z_Q = 0.17\ a$, $R = 0.90\ a$.

Changing the signs of both U and α merely reverses all velocities and the signs of ϕ and ψ. To reverse the solid end for end, the x-axis may be drawn in the opposite direction. (See Reference 2, Section 15.24; Reference 7, p.61.)

125. AIRSHIP FORMS

Any combination of sources and sinks immersed in a uniform stream, as in the last two cases, gives rise to a dividing surface which separates the fluid in the stream from that belonging to the sources and sinks. This surface can be taken as the surface of a solid body, and the formulas for the combined field then represent streaming flow past this body; or it may be the surface of a shell containing within it the sources and sinks. In the latter case, the introduction into the mathematical formulas of terms representing a uniform stream serves merely to procure satisfaction of the boundary condition on the shell.

The dividing surface will be of finite extent provided the total strengths of sources and sinks are equal. Otherwise it will extend to infinity, in the direction of the stream if sources predominate, so that fluid must be carried away on the whole, or in the opposite direction if sinks predominate.

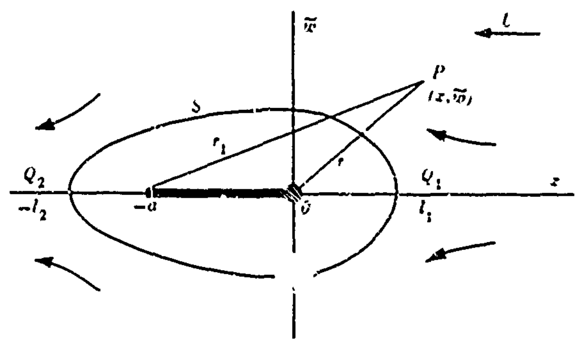


Figure 208 – Diagram for a source at 0 and a compensating line of sinks, in a stream. See Section 125.

As a further example, suppose that sinks are distributed continuously and uniformly along the x-axis from $-a$ to 0 and that there is also a single source at the origin of strength numerically equal to the total strength of the sinks, together with a superposed uniform flow at velocity U toward negative x; see Figure 208.

Let $-4\pi\alpha$ denote the volume of fluid absorbed by the sinks on unit length of the axis; α is thus a negative number and represents algebraically the source density on the axis. Then, if the volume emitted per second by the single source is $4\pi A$, $4\pi A = -4\pi a\alpha$ and

$$\alpha = -\frac{A}{a}. \qquad [125a]$$

The total potential ϕ and stream function ψ at any point P or (x, ϖ), where ϖ denotes distance from the x-axis, can be written, from [119a,b,d,f] and [123d,e],

$$\phi = U\left[x + b^2\left(\frac{1}{r} - \frac{1}{a}\ln\frac{r_1 + x + a}{r + x}\right)\right] = U\left[x + b^2\left(\frac{1}{r} - \frac{1}{a}\ln\frac{r + r_1 + a}{r + r_1 - a}\right)\right] \qquad [125b]$$

$$\psi = U\left[\frac{1}{2}\varpi^2 + b^2\left(\frac{x}{r} - \frac{r_1 - r}{a}\right)\right] = U\left[\frac{1}{2}\varpi^2 + b^2\,\frac{(r_1 - r)^2 - a^2}{2ar}\right] \qquad [125c]$$

where $b^2 = A/U$ and r_1, r are the distances of P from the two ends of the line of sinks or, as illustrated in Figure 208,

$$r = (x^2 + \varpi^2)^{1/2}, \quad r_1 = [(x + a)^2 + \varpi^2]^{1/2}$$

The x-axis is a line of symmetry. The components of velocity are

$$q_x = U\left\{-1 + b^2\left[\frac{x}{r^3} - \frac{1}{a}\left(\frac{1}{r} - \frac{1}{r_1}\right)\right]\right\} \qquad [125d]$$

$$q_{\varpi} = b^2 U\left[\frac{\varpi}{r^3} - \frac{1}{a\varpi}\left(\frac{x+a}{r_1} - \frac{x}{r}\right)\right] \qquad [125e]$$

Compare [123f,g].

On the positive x-axis $r = x$, $r_1 = x + a$, and

$$q_x = U\left(-1 + \frac{ab^2}{x^2(x+a)}\right) \qquad [125f]$$

Thus a stagnation point Q_1 occurs on the x-axis at $x = l_1$ where

$$l_1^2(l_1 + a) = ab^2 \qquad [125g]$$

Again, where $x < -a$, $r = -x$, $r_1 = -x - a$,
and

$$q_x = -U\left(1 + \frac{ab^2}{x^2(x+a)}\right) \qquad [125h]$$

Thus a second stagnation point Q_2 occurs at $x = -l_2$ where $l_2 > a$
and

$$l_2^2(l_2 - a) = ab^2 \qquad [125i]$$

On the x-axis, except on the segment from $-a$ to 0, $\psi = 0$. Furthermore, $\psi = 0$ on the surface S whose equation is

$$\varpi^2 = 2b^2\left(\frac{r_1 - r}{a} - \frac{x}{r}\right) = b^2\,\frac{a^2 - (r_1 - r)^2}{ar} \qquad [125j]$$

By expanding r_1 and r_2 in powers of ϖ, using the binomial theorem, and then dividing through by ϖ^2, it can be verified that S has rounded ends at Q_1 and Q_2. Its shape depends only on the value of b/a, while its size is proportional to b, since the equation remains satisfied when b and all linear dimensions including a are changed in the same ratio. The outline becomes more slender, especially toward the rear, as b/a is diminished.

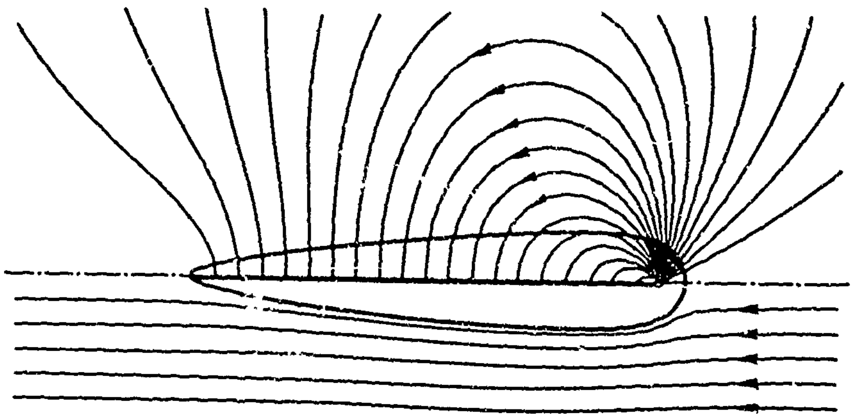


Figure 209 – Airship form constructed with use of a source and a line of sinks. Streamlines due to the source and sinks alone are shown above, resultant streamlines below. See Section 125. (Copied from Reference 8.)

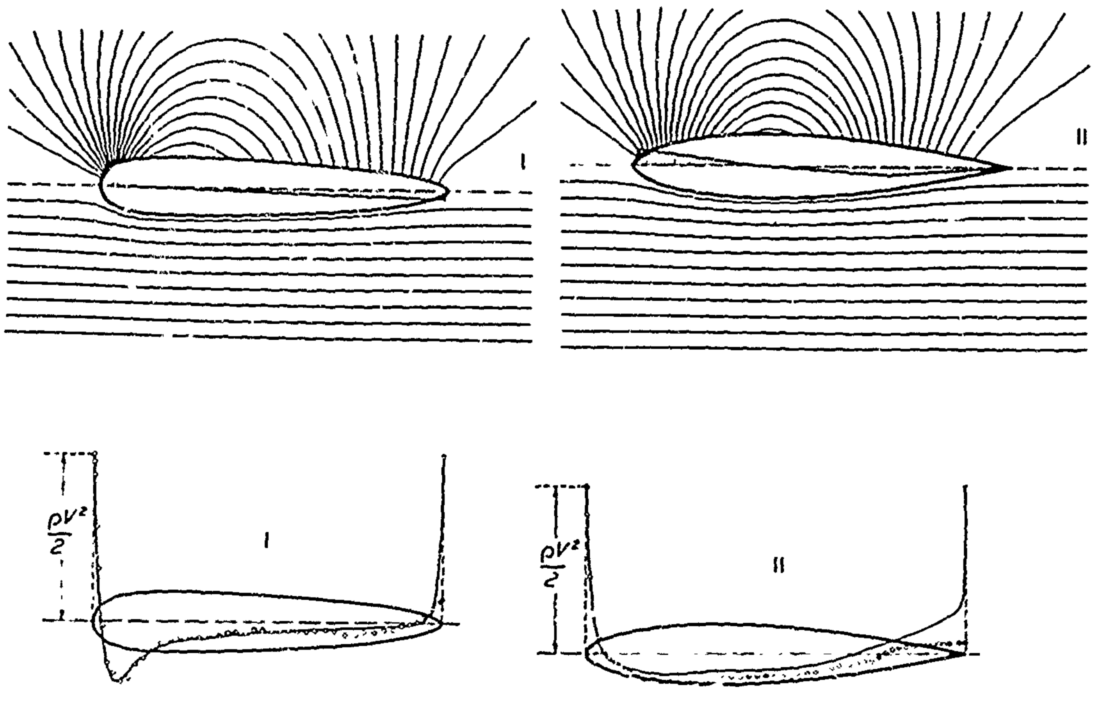


Figure 210 – Two airship forms constructed by Fuhrmann.

In the upper two figures the assumed distribution of sources and sinks is plotted along the axis, and streamlines due to them alone are shown above the axis; resultant streamlines in the flow past the solid are shown below the axis. In the lower figures the calculated pressure distribution is shown by a solid curve in comparison with pressures as observed on a model in air, represented by small circles. The flow is from the left. See Section 125. (Copied from Reference 133.)

Some of the streamlines on one side are shown in Figure 209, past the lower side of a body with $b/a = 0.054$.

Changing the sign of U in the formulas changes sources into sinks and vice versa, and reverses all velocities.

To reverse the solid lengthwise, the x-axis may be drawn in the opposite direction.

Other shapes can be obtained by using various distributions of sources and sinks along the axis. It is not possible to produce in this manner any given arbitrary shape, but many practical airship forms can be imitated closely by dividing the axis into segments and assuming the proper source strength on each segment. Graphical methods for this purpose were discussed by Weinig.[228]

Two shapes thus obtained by Fuhrmann[229] are shown in Figure 210. (See Reference 1. Article 97; Reference 2, Section 15.25; Reference 7, page 633; Reference 229.

126. SPACE DISTRIBUTIONS OF POINT SOURCES

The flow due to any assigned distribution of point sources can be found by integration of the formulas for a single point source. The potential is mathematically identical with the electrostatic potential due to a corresponding distribution of electrical charges in empty space; each unit charge in the electrical problem represents an emission of 4π units of volume per second in the hydrodynamical problem.

The potential due to an axially symmetric distribution of sources on a plane can be expressed in terms of Bessel functions. See Reference 1, Article 102, where the particular cases of a uniform distribution over a circular area and of a distribution proportional to $(a^2 - r^2)^{-1/2}$ are treated.

127. TRANSLATION OF A SPHERE IN INFINITE FLUID

Consider a sphere of radius a moving at velocity U through fluid that is at rest at infinity, as illustrated in Figure 211. The boundary condition to be satisfied at the surface of the sphere is that the fluid and the sphere must have a common component of velocity normal to the surface. The magnitude of this component is $U \cos \theta$ in terms of angular position on the sphere measured from a radius drawn in the direction of motion.

A known type of flow in which the radial velocity varies as $\cos \theta$ and in which the velocity vanishes at infinity is that of a point dipole. The radial velocity due to a dipole located at the center of the sphere can be written, in terms of its moment μ, as in [119h],

$$q_r = 2\mu \frac{\cos \theta}{r^3}$$

At $r = a$ this equals $U \cos \theta$ for all values of θ provided $\mu = a^3 U/2$.

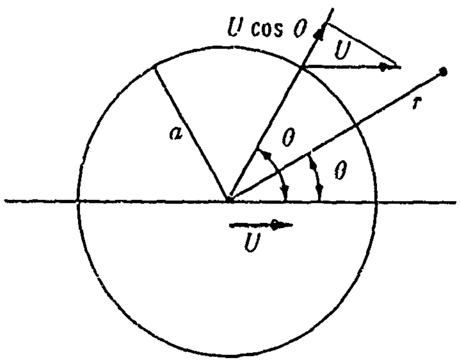


Figure 211 – A sphere in translation. See Section 127.

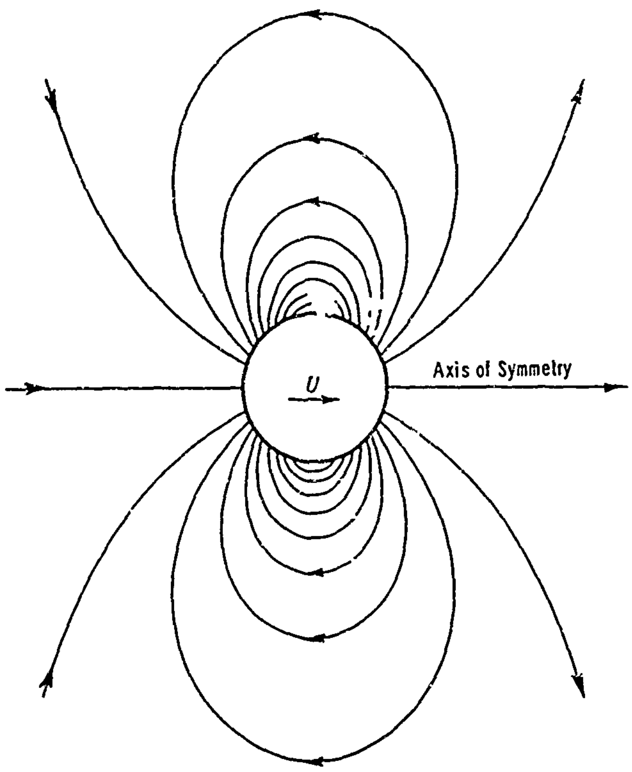


Figure 212 – Streamlines due to a moving sphere. (Copied from Reference 1.)

Using also Equation [119g], the potential ϕ and stream function ψ in the fluid as thus found are

$$\phi = \frac{a^3 U}{2} \frac{\cos\theta}{r^2}, \quad \psi = -\frac{a^3 U}{2} \frac{\sin^2\theta}{r} \qquad [127a,b]$$

The radial and tangential components of velocity are

$$q_r = -\frac{\partial\phi}{\partial r} = a^3 U \frac{\cos\theta}{r^3}, \quad q_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{a^3 U}{2} \frac{\sin\theta}{r^3}; \qquad [127c,d]$$

and

$$q^2 = q_r^2 + q_\theta^2 = \frac{a^6 U^2}{r^6} \left(1 - \frac{3}{4} \sin^2 \theta\right) \qquad [127e]$$

On the sphere $q^2 = U^2(1 - 3/4 \sin^2 \theta)$, so that q has the minimum value $U/2$ at $\theta = \pi/2$, where the fluid is moving directly backward. At front and rear $q = U$.

The streamlines are illustrated for equally spaced values of ψ in Figure 212.

The kinetic energy of the fluid, if its density is ρ, is

$$T = \frac{1}{2} \rho \int_a^\infty r^2 \, dr \int_0^\pi q^2 \cdot 2\pi \sin \theta \, d\theta = \frac{\pi}{3} \rho \, a^3 U^2 \qquad [127f]$$

Here, because of the symmetry, a ring-shaped element of volume, represented by $2\pi r^2 \sin \theta d\theta dr$, has been employed. (See Reference 1, Articles 92, 96; Reference 2, Section 15.32.)

128. STREAMING FLOW PAST A SPHERE

The flow around a stationary sphere when the fluid at infinity has a uniform velocity U is obtained from the results of the last section by imparting to everything an additional uniform velocity $-U$. With appropriate terms added from [119a,b], in which $r \cos \theta$ and $r \sin \theta$ replace x and ϖ, the total potential and stream function are

$$\phi = U\left(r + \frac{a^3}{2r^2}\right) \cos \theta, \quad \psi = \frac{1}{2} U \left(r^2 - \frac{a^3}{r}\right) \sin^2 \theta \qquad [128a,b]$$

Here, for $U > 0$, the flow at infinity is toward $\theta = \pi$.

Thus $\psi = 0$ when $\theta = 0$ or $\theta = \pi$, and also if $r = a$. This shows that a streamline approaches along the radius $\theta = 0$, divides, passes around the sphere, reunites and continues along the radius $\theta = \pi$.

The velocity components of interest are

$$q_r = -U\left(1 - \frac{a^3}{r^3}\right) \cos \theta, \quad q_\theta = U\left(1 + \frac{a^3}{2r^3}\right) \sin \theta \qquad [128c,d]$$

On the sphere, where $r = a$, $q = |q_\theta| = 3/2 \, |U| \sin \theta$, so that q has a maximum value of $3U/2$ on the equator at $\theta = \pi/2$. Stagnation points occur at $\theta = 0$ and $\theta = \pi$.

If the motion is steady, the pressure on the sphere, by the Bernoulli equation, is

$$p = \frac{1}{2}\rho\ (U^2 - q^2) + p_\infty = \frac{1}{2}\rho U^2 \left(1 - \frac{9}{4}\sin^2\theta\right) + p_\infty \qquad [128e]$$

where p_∞ is the pressure at infinity

Thus $p = p_\infty$ at $\theta = 41° 49'$ and at $138° 11'$. On the radii $\theta = 0$ or $\theta = \pi$, $q = |q_r|$ and

$$p = \frac{1}{2}\rho\ U^2 \left(2\frac{a^3}{r^3} - \frac{a^6}{r^6}\right) + p_\infty \qquad [128f]$$

In the equatorial plane, where $\theta = \pi/2$, $q = |q_\theta|$ and

$$p = -\frac{1}{2}\rho\ U^2 \left(\frac{a^3}{r^3} + \frac{a^6}{4r^6}\right) + p_\infty \qquad [128g]$$

The streamlines are illustrated in Figure 213, on a typical plane through the axis of symmetry. The lines drawn are equally spaced at infinity and differ by equal increments of the quantity $\psi/(r \sin\theta)$. The curves labeled $p - p_\infty$ show on an arbitrary scale the excess of

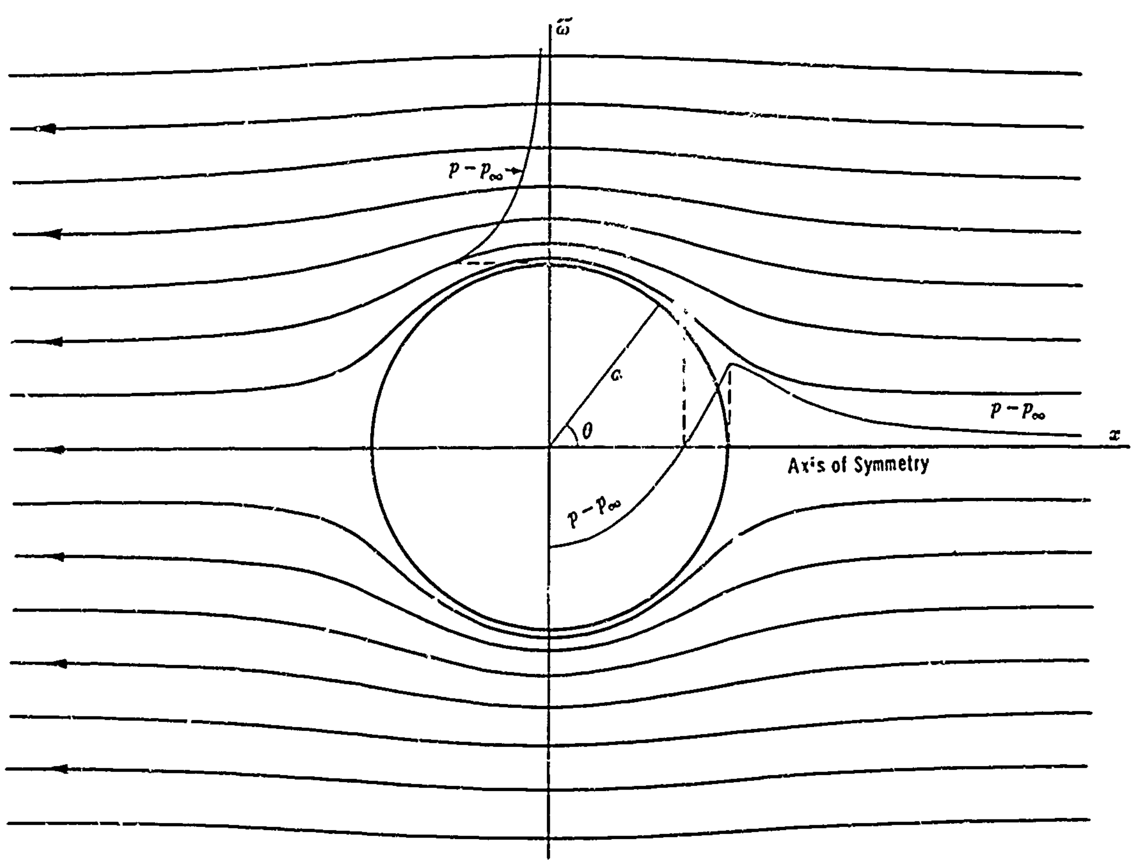


Figure 213 – Streamlines past a sphere, and pressure distribution along the axis of symmetry, over the sphere, and over the equatorial plane. See Section 128.

pressure along the x-axis and over the sphere for positive x, also along the y-axis, where the pressure is plotted horizontally with negative values toward the left. The pressure is symmetrical on front and rear. For this reason it is obvious that there is no resultant force on the sphere.

The sphere can be regarded as a Rankine solid for which the source and sink, while increasing indefinitely in strength, have come together to form a dipole.

The formulas will also represent the flow inside a spherical shell caused by a dipole of moment $\mu = a^3U/2$ at the center. In this case both U and p_∞ represent mathematical constants.

Changing the sign of U reverses all velocities. (See Reference 2, Section 15.30.)

129. SPHERE WITHIN A CONCENTRIC SPHERE

If the moving sphere of Section 126 is surrounded by a fixed concentric spherical shell of radius b, there are two boundary conditions to be satisfied by the field of velocity in the intervening fluid: at $r = a$, $q_r = U \cos \theta$; at $r = b$, $q_r = 0$. In order to have two adjustable constants, let a potential function be assumed of such a form as to represent the superposition of uniform and dipole flow, namely, from Section 128,

$$\phi = \left(U'r + \frac{A}{r^2}\right)\cos \theta, \quad q_r = -\frac{\partial \phi}{\partial r} = \left(-U' + \frac{2A}{r^3}\right)\cos \theta$$

where the angle θ is measured from the direction of motion, and the constants U' and A are to be determined. The boundary conditions require that

$$-U' + \frac{2A}{a^3} = U, \ -U' + \frac{2A}{b^3} = 0$$

Solving for U' and A and adding the stream function ψ from [128b]

$$\phi = \frac{a^3U}{b^3 - a^3}\left(r + \frac{b^3}{2r^2}\right)\cos \theta, \quad \psi = \frac{a^3U}{2(b^3 - a^3)}\left(r^2 - \frac{b^3}{r}\right)\sin^2 \theta \qquad [129a,b]$$

$$q_r = \frac{a^3U}{b^3 - a^3}\left(\frac{b^3}{r^3} - 1\right)\cos \theta, \quad q_\theta = \frac{a^3U}{b^3 - a^3}\left(1 + \frac{b^3}{2r^3}\right)\sin \theta \qquad [129c,d]$$

The possibility of satisfying the boundary conditions in this way for all values of θ arises from the choice of a suitable function for ϕ. The solution is exact, however, only at the instant at which the centers of sphere and shell coincide. Streamlines for equally spaced values of ψ are shown in Figure 214.

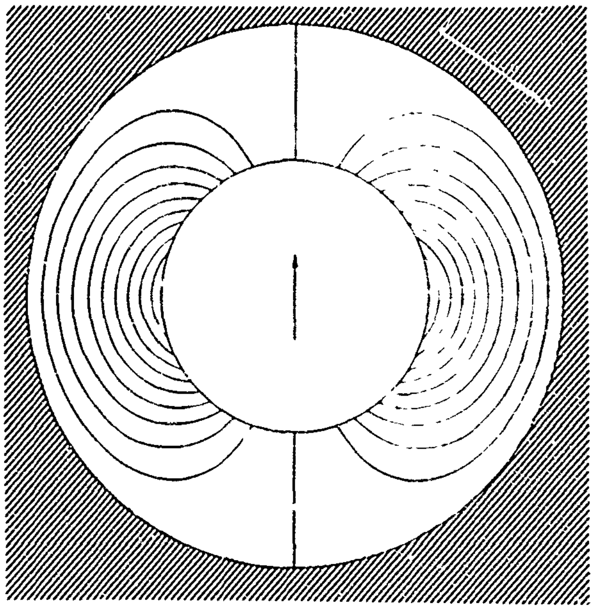


Figure 214 – Streamlines due to a moving sphere momentarily concentric with a surrounding stationary spherical shell. See Section 129.

The kinetic energy of the fluid is, from [17c], in which $q_n = 0$ on the shell and $r = a$ on the sphere,

$$T = \frac{1}{2} \rho \int \phi \, q_r \, dS = \frac{1}{3} \pi \frac{b^3 + 2a^3}{b^3 - a^3} \rho a^3 U^2 \qquad [129e]$$

The integration extends only over the sphere, where $r = a$, $dS = 2\pi a^2 \sin\theta \, d\theta$.

All of these formulas hold momentarily only, as the center of the moving sphere passes the center of the shell. (See Reference 1, Article 93.)

130. SPHERE AND A WALL; TWO SPHERES

The flow caused by a small sphere moving in the presence of a rigid wall can be found to the first order of approximation by elementary methods. Let a be the radius of the sphere and x the distance of its center C from the wall; let it be moving at speed U in a direction inclined at an angle α to a line OCT drawn perpendicularly away from the wall, as shown in Figure 215. Using the method of successive approximation, let three flows be superposed, as follows:

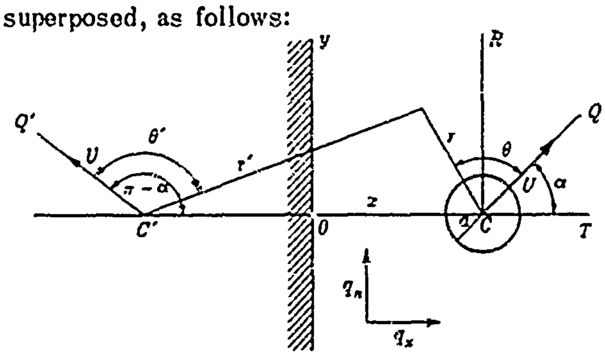


Figure 215 – Diagram for a small sphere near a wall. See Section 130.

1. Assume first the flow caused by the sphere in unbounded fluid. Its potential is that of a dipole at C or, as in Equation [127a],

$$\phi_1 = \frac{a^3 U}{2} \frac{\cos \theta}{r^2}$$

where r denotes distance from C and the angle θ is measured from a line CQ drawn in the direction of U.

2. To satisfy the boundary condition on the wall, add an equal dipole at the mirror image C' of C in the wall, with axis $C'Q'$ in the plane QCT but inclined at an angle $\pi - \alpha$ to OT produced backward; see Figure 215. The potential of this dipole is, similarly,

$$\phi_1' = \frac{a^3 U}{2} \frac{\cos \theta'}{r'^2}$$

where θ' is measured from $C'Q'$.

3. At the sphere, since a/x is small, the partial flow due to ϕ_1' is practically uniform, with components of velocity u in the direction CT and v in a perpendicular direction CR lying in the plane QCT where, from [127c,d], in which $r = r' = 2x$ and $\theta = \theta' = \pi - \alpha$,

$$u = q_{r'} = - a^3 U \frac{\cos \alpha}{8 x^3}, \qquad v = - q_{\theta'} = - \frac{a^3 U}{2} \frac{\sin \alpha}{8 x^3}$$

These two uniform components of flow, in interaction with the sphere, add a potential that may be obtained from [128a] by first replacing U by $-u$ and θ by θ_1, where θ_1 is measured from CT, then replacing U by $-v$ and θ by θ_2, where θ_2 is measured from CR, and adding the results. The value of this potential will be needed only on the surface of the sphere, where $r = a$. There its value is

$$\phi_2 = \frac{a^3 U}{8x^3} \frac{3a}{2} \left(\cos \theta_1 \cos \alpha + \frac{1}{2} \cos \theta_2 \sin \alpha \right)$$

The potential ϕ_2 includes the variable part of ϕ_1'; a constant part equal to the value of ϕ_1' at the center C has been omitted, but this omission has no effect upon the result to be obtained.

Now introduce also an angle ω, so that r, θ, ω are polar coordinates with origin at C; let ω be measured from the plane QCT; see Figure 216, where P is any point on the unit sphere about C. By projection of CP on CT and CR it is seen that

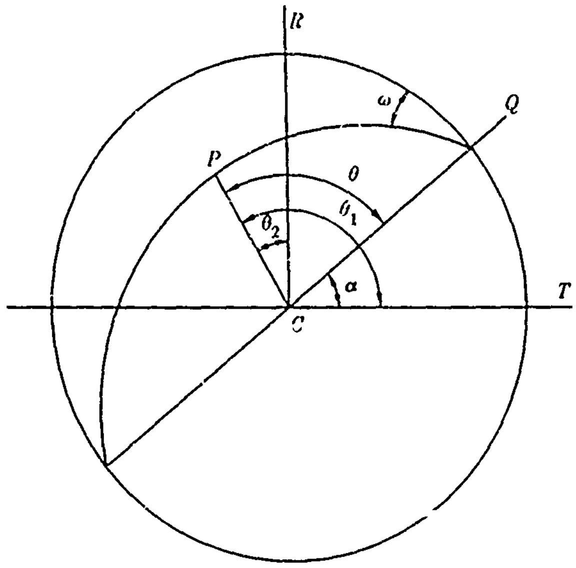


Figure 216 – Polar coordinates for a small sphere near a wall.

$$\cos \theta_1 = \cos \theta \ \cos \alpha \ - \sin \theta \ \cos \omega \sin \alpha ;$$
$$\cos \theta_2 = \cos \theta \ \sin \alpha \ + \sin \theta \ \cos \omega \cos \alpha .$$

The kinetic energy T of the fluid may then be found by substituting in Equation [17c] $\phi = \phi_1 + \phi_2$ with $r = a$, also $q_n = U \cos \theta$ and $dS = a^2 \sin \theta \, d\theta \, d\omega$, and integrating over the sphere. It is

$$T = \frac{1}{2} \rho \int \phi \, q_n \, dS = \frac{1}{3} \pi \rho \, a^3 U^2 \left[1 + \frac{3}{16} \frac{a^3}{x^3} \left(1 + \cos^2 \alpha\right) \right]. \qquad [130a]$$

The forces on the sphere may now be found by means of Lagrange's equation,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} = Q,$$

where q stands for any coordinate of the sphere and $\dot{q} = dq/dt$.

In terms of the Cartesian coordinates x and y of the center of the sphere, with velocities $\dot{x} = U \cos \alpha$, $\dot{y} = U \sin \alpha$,

$$T = \frac{1}{3} \pi \rho \, a^3 \left[\left(1 + \frac{3}{8} \frac{a^3}{x^3}\right) \dot{x}^2 + \left(1 + \frac{3}{16} \frac{a^3}{x^3}\right) \dot{y}^2 \right]. \qquad [130b]$$

For x and y as coordinates, the generalized forces Q are simply the ordinary components of the force on the fluid, or $-X$ for x and $-Y$ for y where X and Y are the components of the force exerted by the fluid on the sphere, respectively away from and parallel to the wall.

Taking $q = x$, then $q = y$,

$$\frac{d}{dt}\left[\frac{2}{3}\pi\rho a^3\left(1+\frac{3}{8}\frac{a^3}{x^3}\right)\dot{x}\right]+\frac{3}{8}\pi\rho\frac{a^6}{x^4}\left(\dot{x}^2+\frac{1}{2}\dot{y}^2\right)=-X,$$

$$\frac{d}{dt}\left[\frac{2}{3}\pi\rho a^3\left(1+\frac{3}{16}\frac{a^3}{x^3}\right)\dot{y}\right]=-Y,$$

whence

$$X=-\frac{2\pi}{3}\rho a^3\left(1+\frac{3}{8}\frac{a^3}{x^3}\right)\ddot{x}+\frac{3}{16}\pi\rho U^2\frac{a^6}{x^4}\left(2\cos^2\alpha-\sin^2\alpha\right), \qquad [130c]$$

$$Y=-\frac{2\pi}{3}\rho a^3\left(1+\frac{3}{16}\frac{a^3}{x^3}\right)\ddot{y}+\frac{3}{8}\pi\rho U^2\frac{a^6}{x^4}\sin\alpha\cos\alpha\,. \qquad 130d]$$

Here $\ddot{x}$, $\ddot{y}$ are the components of the acceleration of the sphere. The term in U^2 in X represents a repulsion by the wall on a sphere moving toward or away from it, proportional to $1/x^4$, or an attraction half as large on a sphere moving parallel to it.

(See Reference 1, Articles 98, 99, 137, 138; Reference 2, Section 16.30.)

Two Spheres

Instead of a wall. there may be a similar sphere centered at 0′ and moving at speed U in the direction $\theta' = 0$, so as to secure complete symmetry of motion.

The *general* motion of two spheres of any size can be treated in terms of series of spherical harmonics[1]; the motion has also been treated in terms of images by Hicks,[230] otherwise by Bassett,[5, 231] and in terms of bipolar coordinates by Endo.[232]

131. POINT DIPOLES NEAR A SPHERE

Consider two point dipoles located at $(b_1, 0, 0)$ and $(b_2, 0, 0)$, with their axes parallel to the x-axis but oppositely directed; let their moments be μ_1 μ_2, where μ_1 and μ_2 have opposite signs. The resulting strea.n function, if the fluid is at rest at infinity, is, from Equation [119k],

$$\psi=-\mu_1\frac{\tilde{\omega}^2}{r_1^3}-\mu_2\frac{\tilde{\omega}^2}{r_2^3} \qquad [131a]$$

where

$$\tilde{\omega} = (y^2 + z^2)^{1/2}, \; r_1 = [(x - b_1)^2 + \tilde{\omega}^2]^{1/2}, \; r_2 = [(x - b_2)^2 + \tilde{\omega}^2]^{1/2}. \qquad [131b, c, d]$$

The stream surface for $\psi = 0$ is given by

$$r_2^3/r_1^3 = -\mu_2/\mu_1, \text{ or } r_2^2/r_1^2 = k$$

where

$$k = (-\mu_2/\mu_1)^{2/3} > 0,$$

or also, after replacing r_1 and r_2 by their equivalents,

$$(x - b_2)^2 + \tilde{\omega}^2 = k\,[(x - b_1)^2 + \tilde{\omega}^2]$$

or

$$(1 - k)(x^2 + \tilde{\omega}^2) + 2\,(k\,b_1 - b_2)\,x = k\,b_1^2 - b_2^2.$$

This is the equation of a sphere.

Let the origin be transferred to its center. Then the term in x disappears from its equation; hence the new values of b_1 and b_2 are such that $k\,b_1 = b_2$, and the radius a of the sphere is given by

$$a^2 = \frac{k\,b_1^2 - b_2^2}{1 - k} = k\,b_1^2 = b_1\,b_2. \qquad [131e]$$

Thus the dipoles are located at inverse points with respect to the sphere; see Figure 217 in which two alternative cases are illustrated. Either b_1 or b_2 must exceed a.

The formulas may represent either, if $b_1 > a$, the flow around a sphere of radius a caused by a dipole of moment μ_1 placed at a distance b_1 from the center of the sphere and with its axis directed radially, or, if $b_1 < a$, the flow inside a spherical shell of radius a caused by a dipole similarly placed inside it. In either case the x-axis is to be drawn from the center through the location of the dipole; and the second dipole, at a distance $b_2 = a^2/b_1$ from the center, becomes a fictitious one that can be regarded as the image of the first. If $\mu_1 > 0$, the axis of the dipole is directed outward from the center. The potential and stream function are

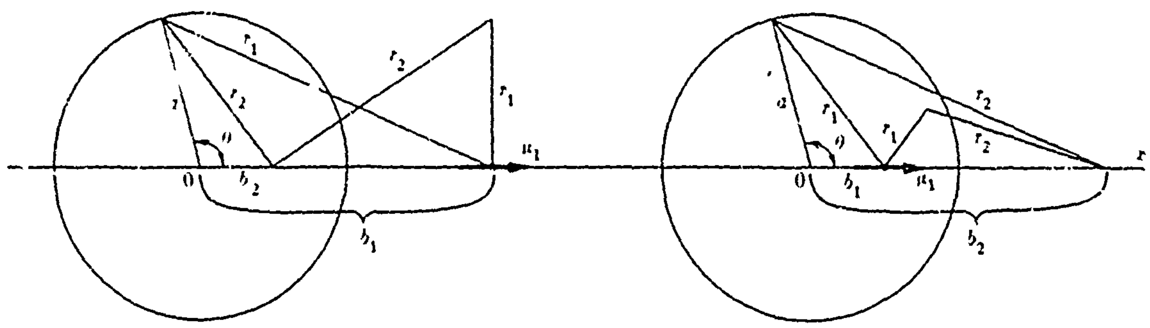


Figure 217 – Point dipole μ_1 either outside or inside of a fixed spherical surface. See Section 131.

$$\phi = \mu_1 \left[\frac{x - b_1}{r_1^3} - \left(\frac{a}{b_1}\right)^3 \frac{x - b_2}{r_2^3} \right], \quad \psi = - \mu_1 \varpi^2 \left[\frac{1}{r_1^3} - \left(\frac{a}{b_1}\right)^3 \frac{1}{r_2^3} \right]. \qquad [131f, g]$$

Here ψ is given by Equation [131a], with k taken from Equation [131e], and ϕ is found by comparison with Equations [119k] and [119o].

The velocity at any point can be found by adding vectorially the velocities due to the two dipoles.

On the sphere itself $r_2/r_1 = a/b_1$ by similar triangles, hence $\psi = 0$ and

$$\phi = \frac{a^2 - b_1^2}{b_1} \frac{\mu_1}{r_1^3}, \quad r_1 = (a^2 + b_1^2 - 2\, a b_1 \cos\theta)^{1/2},$$

where θ is the polar angle at the center measured from a line through the dipole. Thus on the sphere

$$q_\theta = - \frac{1}{a} \frac{\partial \phi}{\partial \theta} = 3\,(a^2 - b_1^2)\,\mu_1 \frac{\sin\theta}{r_1^5}\,; \qquad [131h]$$

and, since the velocity is tangential to the sphere, $q = |q_\theta|$.

If the dipole is at the *center of a spherical shell*, however, a limiting form of ϕ is needed; for, as $b_1 \to 0$, $b_2 \to \infty$. Expanding by the binomial theorem in descending powers of b_2,

$$r_2^{-3} = (b_2^2 - 2\,b_2\,x + x^2 + \varpi^2)^{-3/2} = (b_2^2)^{-3/2} - \frac{3}{2}\,(b_2^2)^{-5/2}\,x\,.....$$

$$(-2\,b_2\,x + x^2 + \varpi^2)\,.... = b_2^{-3} + 3\,b_2^{-4}\,x + ...,$$

hence

$$b_1^{-3}\,(x - b_2)\,r_2^{-3} = -\,a^{-6}\,b_2 - 2\,a^{-6}\,x\,.........$$

since $b_1\,b_2 = a^2$. The term $-a^{-6}\,b_2$ contributes a constant term in ϕ and may be omitted. Then, as $b_1 \to 0$, Equations [131f, g] become

$$\phi = \mu_1\left(\frac{1}{r^2} + \frac{2r}{a^3}\right)\cos\theta,\quad \psi = -\,\mu_1\,\varpi^2\left(\frac{1}{r^3} - \frac{1}{a^3}\right) \qquad [131i, j]$$

with use of polar coordinates at the center of the sphere such that $r = (x^2 + \varpi^2)^{1/2}$, $x = r\cos\theta$. All other terms in the series vanish as $b_2 \to \infty$. The components of velocity are

$$q_r = -\frac{\partial\phi}{\partial r} = 2\,\mu_1\left(\frac{1}{r^3} - \frac{1}{a^3}\right)\cos\theta,\quad q_\theta = -\frac{1}{r}\,\frac{\partial\phi}{\partial\theta} = \mu_1\left(\frac{1}{r^3} + \frac{2}{a^3}\right)\sin\theta. \qquad [131k, l]$$

Thus the presence of the shell superposes a uniform backward flow, with components of velocity $-2\,\mu_1\cos\theta/a^3$ and $2\,\mu_1\sin\theta/a^3$, upon the flow due to the dipole alone.

The *force* F on the sphere or shell is of magnitude

$$|F| = \frac{1}{2}\,\rho\int q^2\cos\theta\,(2\pi a^2\sin\theta d\theta) = \frac{24\pi\rho\,a^3\,b_1\,\mu_1^2}{(b_1^2 - a^2)^4}\,. \qquad [131m]$$

(The integration is long but easy.) The force tends to draw the nearest part of the sphere or spherical shell toward the dipole.

Streamlines for equally spaced values of ψ, on a typical half-plane through the axis of symmetry or x-axis, are shown for an exterior dipole as solid lines in Figure 218.

(See Reference 1, Article 96; Reference 2, Section 15.43.)

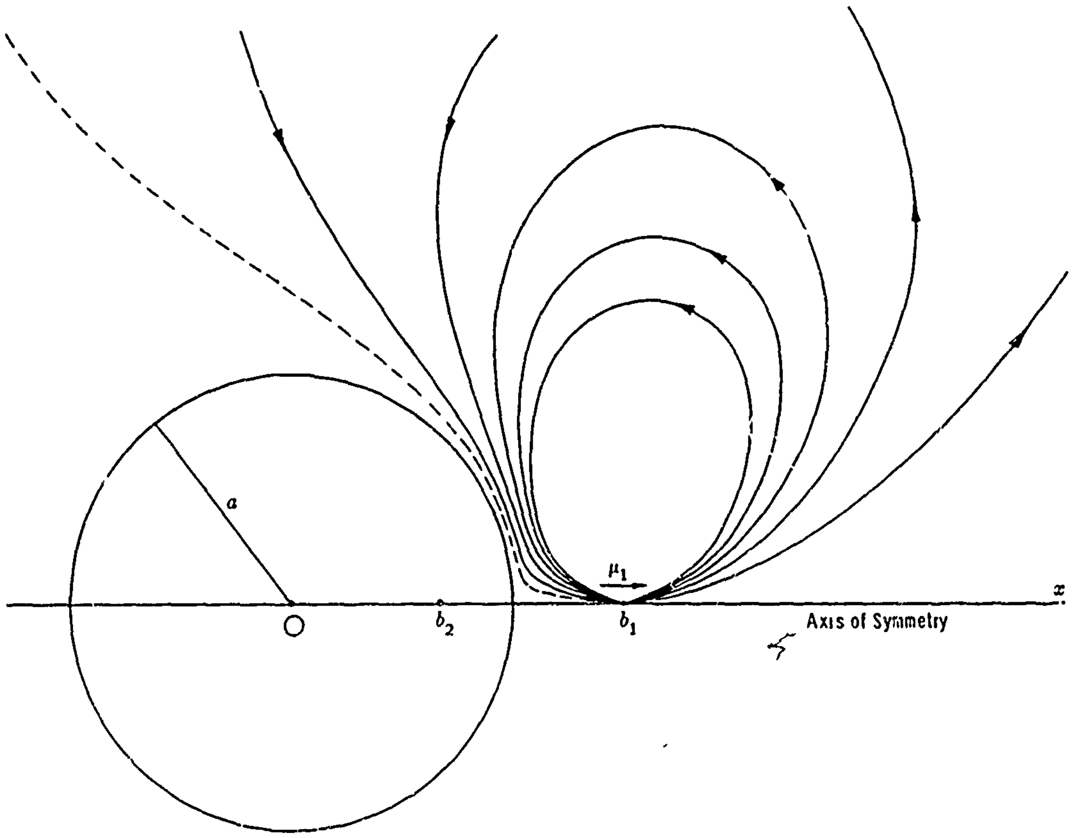


Figure 21?. – Streamlines due to a point dipole at b_1 near a sphere. See Section 131.

132. LINE OF TRANSVERSE DIPOLES

Suppose that point dipoles are distributed continuously along a line with their axes lying perpendicularly to the line but in a common plane. Let the x-axis be taken along the line and the y-axis in the direction of the dipole axes, and let the dipole strength per unit length be ν. Then the potential due to the dipoles on an element dx' located at $x = x'$, as found by writing in Equation [119s] $\mu = \nu\, dx'$, $x_1 = x'$, $y_1 = z_1 = 0$, $l = n = 0$ and $m = 1$, is

$$d\phi = \frac{\nu y\, dx'}{[\varpi^2 + (x - x')^2]^{3/2}}, \quad \varpi = (y^2 + z^2)^{1/2}. \qquad [132a, b]$$

The total potential at (x, y, z) is then $\int d\phi$ which can be evaluated when ν is known as a function of x.

If ν is constant between $x = c_1$ and $x = c_2$ and zero elsewhere,

$$\phi = \nu \int_{c_1}^{c_2} \frac{y\,dx'}{[\tilde{\omega}^2 + (x - x')^2]^{3/2}} = -\frac{\nu y}{\tilde{\omega}^2} \frac{x - x'}{[\tilde{\omega}^2 + (x - x')^2]^{1/2}} \Bigg/_{x' = c_1}^{x' = c_2} .$$

Let ω denote an angle about the x-axis measured from the xy-plane, so that x, $\tilde{\omega}$, ω constitute cylindrical coordinates. Then $y = \tilde{\omega} \cos \omega$ and

$$\phi = \frac{\nu}{\tilde{\omega}} (\cos \theta_1 - \cos \theta_2) \cos \omega \qquad [132c]$$

where θ_1 and θ_2 are the angles between the positive x-axis and lines drawn to (x, y, z) from the ends of the line of dipoles; see Figure 219.

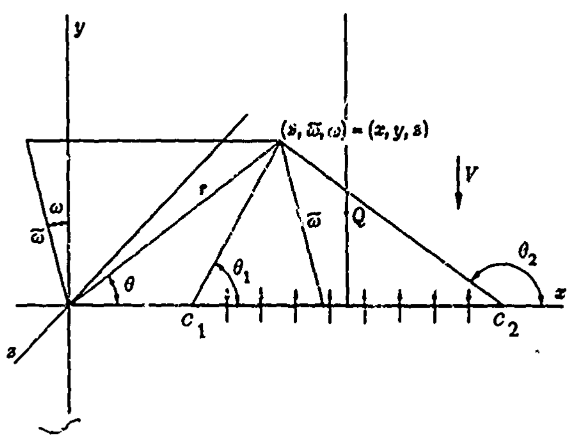


Figure 219 – A line of transverse point dipoles. See Section 132.

If the dipoles extend over the entire x-axis, $c_1 \to -\infty$, $c_2 \to +\infty$, $\theta_1 = 0$, $\theta_2 = \pi$, and

$$\phi = \frac{2\nu}{\tilde{\omega}} \cos \omega = \frac{2\nu y}{\tilde{\omega}^2} \qquad [132d]$$

as for a uniform line dipole; see Sections 15 and 37.

If the dipoles extend only over the positive x-axis, $\theta_2 = \pi$ but $\theta_1 = \theta$ where θ is measured from the x-axis to a radius drawn from the origin, and

$$\phi = \frac{\nu}{\tilde{\omega}} (1 + \cos \theta) \cos \omega = \frac{\nu}{\tilde{\omega}} \left(1 + \frac{x}{r}\right) \cos \omega \qquad [132e]$$

where $r = (\tilde{\omega}^2 + x^2)^{1/2}$; or, if r, θ, and ω are employed as polar coordinates,

$$\phi = \frac{\nu}{r} \frac{1 + \cos\theta}{\sin\theta} \cos\omega. \qquad [132f]$$

(See Reference 7, page 70.)

133. TRANSVERSE FLOW PAST A SOLID OF REVOLUTION

Let the uniform line of transverse dipoles on the x-axis as described in the first part of the last section be immersed in a uniform stream having velocity V toward negative y. For the stream, $\phi = Vy = V\tilde{\omega}\cos\omega$, hence the resultant potential is, from Equation [132c],

$$\phi = \left[V\tilde{\omega} + \frac{\nu}{\tilde{\omega}}(\cos\theta_1 - \cos\theta_2)\right]\cos\omega. \qquad [133a]$$

Both the xy- and zx-planes are planes of geometrical symmetry; the equipotential surface for $\phi = -\phi_1$ is the mirror reflection in the zx-plane of that for $\phi = \phi_1$, so that the streamlines are symmetrically disposed. A third plane of symmetry is the bisector of the segment $c_1 c_2$. All of the equipotential surfaces are asymptotic at infinity to planes perpendicular to the y-axis; that for $\phi = 0$ is the zx-plane itself, on which $\omega = \pm\pi/2$.

On any plane through the line of dipoles or the x-axis, the trace of an equipotential surface is a curve defined by

$$V\tilde{\omega} + \frac{\nu}{\tilde{\omega}}(\cos\theta_1 - \cos\theta_2) = \frac{\phi}{\cos\omega} = \text{constant}. \qquad [133b]$$

Clearly the same geometrical set of equipotential curves occurs on all of these planes but the value of ϕ attached to a given curve is proportional to $\cos\omega$.

It suffices, therefore, to study the curves on the xy-plane, where $|\cos\omega| = 1$. Assume that $V > 0$, $\nu > 0$. Then, since $\cos\theta_1 - \cos\theta_2 \geqq 0$, it is clear that, on the part of the plane on which $y > 0$, $\phi \to \infty$ both as $y = \tilde{\omega} \to \infty$ and as $y \to 0$ with x lying between c_1 and c_2, so that $\theta_1 \to 0$, $\theta_2 \to \pi$. Hence, in particular, on the line $x = (c_1 + c_2)/2$, a relative minimum of ϕ must occur at some point Q; see Figure 219. From the character of the flow caused by dipoles, it is clear that the fluid will flow away from Q both toward $x > 0$ and toward $x < 0$, and hence that the potential must decrease in both of these directions. The point Q is thus a saddle point for ϕ, and hence also a stagnation point, since the ω-component of the velocity vanishes by symmetry. On the half-plane where $y < 0$, symmetrical relations occur.

The equipotential curves will thus have the general character of those shown in Figure 220, where the axes are so placed that $c_2 = -c_1$. The broken lines show the equipotentials that meet at Q.

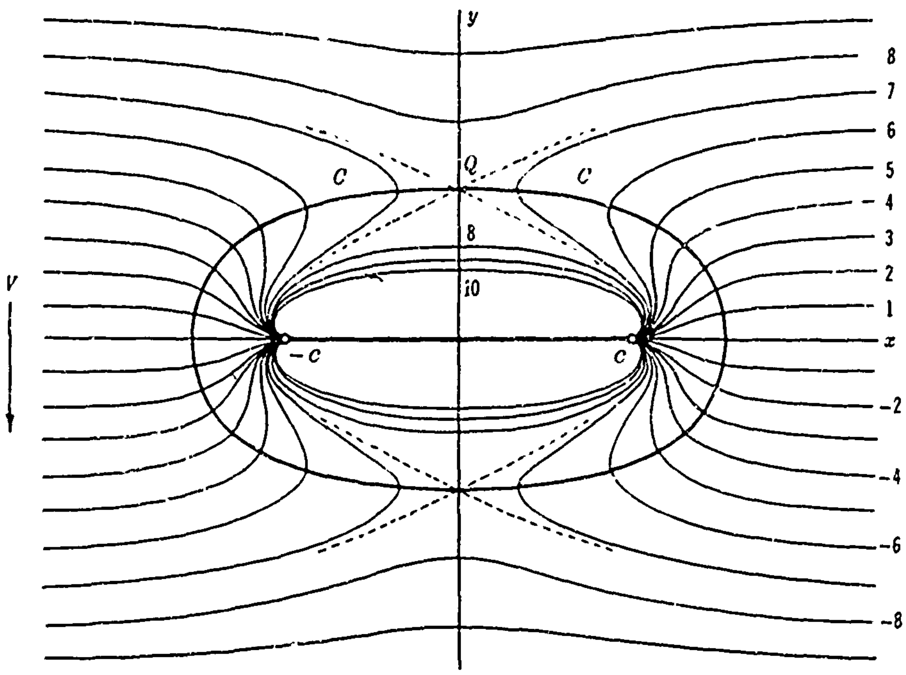


Figure 220 – For a uniform line of transverse point dipoles in a stream flowing perpendicularly to the line, traces of the equipotential surfaces are shown on a plane drawn through the line of dipoles and parallel to the stream. The equipotential surfaces may be those of the transverse flow past a certain solid of revolution which is represented, in a section through its axis of symmetry, by the heavy closed curve. See Section 133. (Copied from Reference 7.)

The streamlines will be three-dimensional, in general, but in the xy-plane they will be plane curves orthogonal to the equipotential curves. Clearly there will be one streamline which, approaching with y decreasing, divides at Q and passes around the line of dipoles along a closed dividing curve C, then re-unites at the other stagnation point and proceeds to $y = -\infty$. The same geometrical curve C can be drawn on any plane through the x-axis: and on all planes it will have the property that at any point the component of the velocity lying in the plane will be tangent to the curve, since both this component and the curve C must be perpendicular to the equipotential curve through the point. The surface of revolution generated by rotation of the curve C about the x-axis is thus a dividing surface and may be taken as the surface of a solid body. The formulas then represent the transverse flow past this body, or, also, the flow caused by the line of dipoles inside of a similar shell.

An example of the equipotential curves in the xy-plane is shown in Figure 220, where $c_1 = c$, $c_2 = -c$ and the heavy curve is C or the trace of the dividing surface of revolution.

Further mathematical details will be given only for the case in which the line of dipoles extends to infinity in one direction.

Half-Infinite Solid of Revolution

Let the dipoles extend from $x = 0$ to $x \to \infty$. Then $\theta_1 = \theta$, where θ is the polar angle at the origin, $\cos \theta_2 = -1$ and

$$\phi = \left[V \varpi + \frac{\nu}{\varpi} (1 + \cos \theta) \right] \cos \omega. \tag{133c}$$

Here $\cos \theta = x/r$, $r = (\varpi^2 + x^2)^{1/2}$, $\varpi = r \sin \theta$. Hence,

$$q_x = -\frac{\partial \phi}{\partial x} = -\frac{\nu \varpi}{r^3} \cos \omega, \tag{133d}$$

$$q_{\varpi} = -\frac{\partial \phi}{\partial \varpi} = \left[-V + \frac{\nu}{\varpi^2} (1 + \cos \theta + \sin^2 \theta \cos \theta) \right] \cos \omega, \tag{133e}$$

$$q_\omega = -\frac{1}{\varpi} \frac{\partial \phi}{\partial \omega} = \left[V + \frac{\nu}{\varpi^2} (1 + \cos \theta) \right] \sin \omega. \tag{133f}$$

or

$$q_r = -\frac{\partial \phi}{\partial r} = \left(-V \sin \theta + \frac{\nu}{r^2} \frac{1 + \cos \theta}{\sin \theta} \right) \cos \omega, \tag{133g}$$

$$q_\theta = -\frac{1}{r} \frac{\partial \phi}{\partial \theta} = \left(-V \cos \theta + \frac{\nu}{r^2} \frac{1 + \cos \theta}{\sin^2 \theta} \right) \cos \omega, \tag{133h}$$

$$q_\omega = -\frac{1}{r \sin \theta} \frac{\partial \phi}{\partial \omega} = \left(V + \frac{\nu}{r^2} \frac{1 + \cos \theta}{\sin^2 \theta} \right) \sin \omega. \tag{133i}$$

The speed q may be found from $q^2 = q_x^2 + q_{\varpi}^2 + q_\omega^2 = q_r^2 + q_\theta^2 + q_\omega^2$.

As $x \to \infty$, $r \to \infty$, $\theta \to 0$; hence $q_x \to 0$, and, at $\varpi = \sqrt{2\nu/V}$, $q_{\varpi} = 0$ while $q_\omega = 2V \sin \omega$. The streamlines are thus tangent at $x = +\infty$ to a cylinder of diameter $2\sqrt{2\nu/V}$. The solid of revolution must, therefore, be asymptotic to this cylinder.

Some of the equipotential curves and streamlines in the xy-plane are shown in Figure 221; the heavy curve is the outline of the solid. The equipotential curves are extended inward toward the line of dipoles. (In this figure V is denoted by U.)

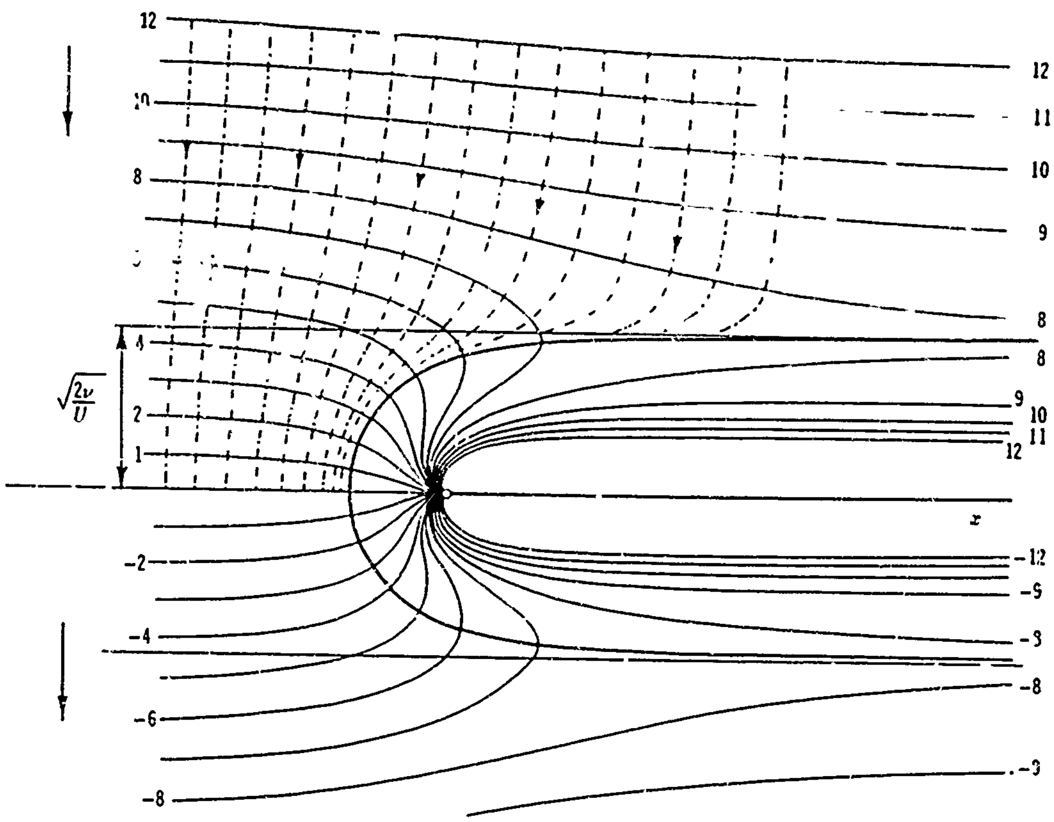


Figure 221 – Similar to Figure 220, but the line of dipoles and the solid of revolution extend to infinity. A few streamlines outside the solid are shown by broken curves. See Section 133. (Copied from Reference 7.)

An attempt to produce forms resembling actual airship hulls more closely was made by Lotz,[233] using a nonuniform distribution of dipoles on the axis, but point sources spread over the surface were found to work better. (See Reference 7, page 69.)

134. POINT SOURCE NEAR A SPHERE

Problems involving given boundary conditions can often be solved by superposing solutions satisfying the given conditions. Sometimes the process of superposition involves an integration.

The field of a point source, for example, is easily obtained by integrating that of a dipole. This corresponds to the fact that physical dipoles, each consisting of a source and equal sink close together, can be laid out in a row so that each source is canceled by a superposed sink, except at the ends of the row.

In this way a dipole located near a sphere, as in Section 131, can be replaced by a point source. In the formulas of Section 131 let b_1 be replaced by x_1' and μ_1 by $-A\,dx_1'$ where A is a constant. Then the flow is represented due to a dipole of moment $-A\,dx_1'$, located on the element dx_1' of the x'-axis at the position x_1'. Let such dipoles be located on all elements from $x_1' = c_1$ to $+\infty$; and integrate to obtain the flow due to all of them. The potential ϕ and stream function ψ thus obtained are, from Equations [131f, g],

$$\phi = -A \int_{c_1}^{\infty} \left[\frac{x - x_1'}{r_1'^3} - \left(\frac{a}{x_1'}\right)^3 \frac{x - x_2'}{r_2'^3} \right] dx_1',$$

$$\psi = A \int_{c_1}^{\infty} \varpi^2 \left[\frac{1}{r_1'^3} - \left(\frac{a}{x_1'}\right)^3 \frac{1}{r_2'^3} \right] dx_1',$$

where

$$x_2' = a^2/x_1',\; r_1' = [(x - x_1')^2 + \varpi^2]^{1/2},\; r_2' = [(x - x_2')^2 + \varpi^2]^{1/2}.$$

Here x and ϖ are the coordinates of a fixed point in space and are constant in the integration; see Figure 222.

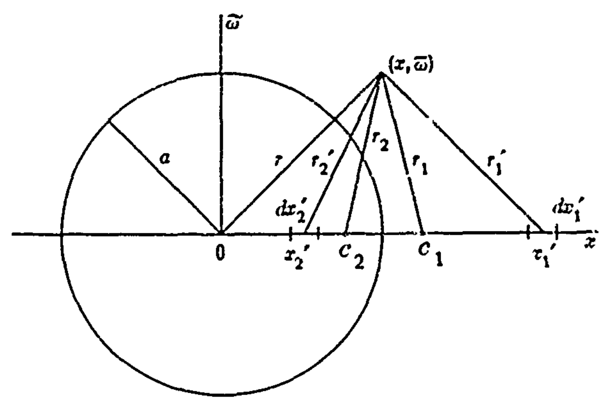


Figure 222 – See Section 134.

The first term of the integral for ϕ can be evaluated at once. In the second term take x_2' as the variable of integration with limits 0 and c_2 where

$$c_2 = a^2/c_1, \tag{134a}$$

and integrate once by parts. Then

$$\phi = A\left(\frac{1}{r_1} + \frac{a}{c_1}\,\frac{1}{r_2} - \frac{1}{a}\int_0^{c_2}\frac{dx_2'}{r_2'}\right) \qquad [134b]$$

where

$$r_1 = [(x - c_1)^2 + \varpi^2]^{1/2},\ r_2 = [(x - c_2)^2 + \varpi^2]^{1/2}.$$

This is the potential due to a point source of strength A located on the x-axis at $x = c_1$, together with that due to another source of strength aA/c_1 at $x = c_2$ and a line of sinks of uniform strength $-A/a$ per unit length extending from the origin to the point $x = c_2$. The second source and the line of sinks may be regarded as the image of the first source in the rigid sphere of radius a. It must be assumed that $c_1 > a$.

Evaluating the last integral, and treating ψ similarly and dropping a useless constant term,

$$\phi = A\left(\frac{1}{r_1} + \frac{a}{c_1}\,\frac{1}{r_2} - \frac{1}{a}\ln\frac{r + x}{r_2 + x - c_2}\right) = A\left(\frac{1}{r_1} + \frac{a}{c_1}\,\frac{1}{r_2} - \frac{1}{a}\ln\frac{r + r_2 + c_2}{r + r_2 - c_2}\right), \qquad [134c]$$

$$\psi = A\left(\frac{x - c_1}{r_1} + \frac{a}{c_1}\,\frac{x - c_2}{r_2} - \frac{r - r_2}{a}\right),\ r = \sqrt{x^2 + \varpi^2}. \qquad [134d,e]$$

These formulas represent the flow in fluid that is at rest at infinity, caused by a source outside a fixed sphere of radius a. With the origin at the center of the sphere, the source is on the x-axis at $x = c_1$; $\varpi = \sqrt{y^2 + z^2}$ and denotes distance from the x-axis. If $A < 0$, the source becomes a sink and all velocities are reversed.

On the sphere itself $r = a$, $r_1/r_2 = a/c_2 = c_1/a$ by similar triangles, $x = a\cos\theta$ and

$$r_2 = (a^2 + c_2^2 - 2ac_2\cos\theta)^{1/2}$$

in terms of the polar angle θ measured from a radius drawn toward the source. Hence, on the sphere, $\psi = -A$, which shows that the sphere is a stream surface, and

$$\phi = \frac{A}{a}\left(\frac{2c_2}{r_2} - \ln\frac{a + r_2 + c_2}{a + r_2 - c_2}\right); \qquad [134f]$$

also

$$q_\theta = -\frac{1}{a}\frac{\partial\phi}{\partial\theta} = \frac{2c_2^2 A}{a r_2}\left(\frac{1}{r_2^2} + \frac{1}{(a + r_2)^2 - c_2^2}\right)\sin\theta, \qquad [134g]$$

and the particle speed is $q = |q_\theta|$.

In Figure 223 streamlines are drawn which leave the source in the same plane in directions 22.5 deg apart, and eventually become parallel to these same directions, which are also indicated in the figure. (See Reference 1, Article 96; Reference 2, Section 15.40.)

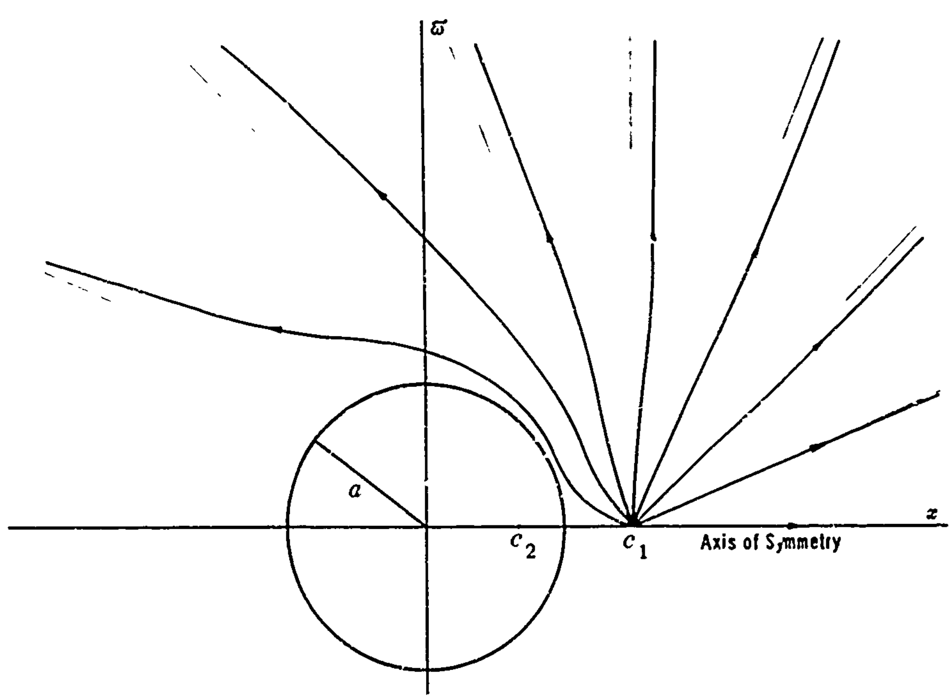


Figure 223 – Streamlines due to a point source at c_1 near a sphere. See Section 134.

135. BOUNDARY CONDITIONS IN ROTATION

The general boundary condition, that the fluid and the boundary must have a common component of velocity normal to the boundary, can be put into a useful special form when the boundary is rotating as a rigid body.

Let the boundary rotate at angular velocity ω_z about the z-axis. Then any point on it located at (x, y, z) is moving parallel to the xy-plane with a linear velocity $\omega_z\,(x^2 + y^2)^{1/2}$; and, by similar triangles, as illustrated in Figure 224, its x and y components of velocity are

$$U = -\omega_z\, y,\; V = \omega_z\, x,\; W = 0$$

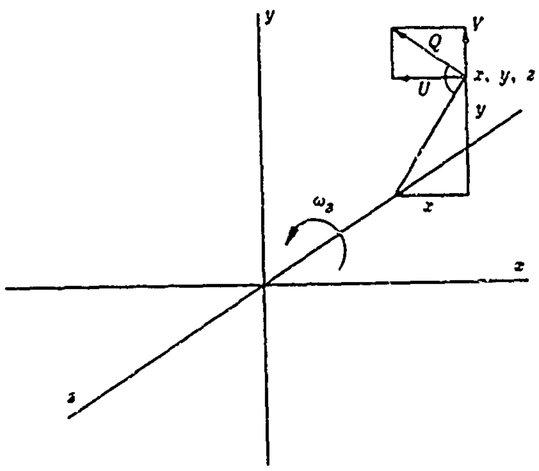


Figure 224 – Velocity due to rotation.

The component Q_n normal to the boundary, at a point on the boundary where the direction cosines of its normal are l, m, n, is then

$$Q_n = lU + mV + nW = \omega_z\ (mx - ly). \qquad [135a]$$

For rotation at velocity ω_x about the x-axis, or ω_y about the y-axis, similarly,

$$Q_n = \omega_x\ (ny - mz),\ Q_n = \omega_y\ (lz - nx). \qquad [135b, c]$$

The three types of rotation may be superposed in order to obtain the most general type of rotation about an axis through the origin.

The normal component of the velocity of the fluid, on the other hand, in terms of its cartesian components u, v, w, is

$$q_n = lu + mv + nw.$$

Equating q_n to Q_n gives as the boundary condition for the most general case

$$lu + mv + nw = \omega_x\ (ny - mz) + \omega_y\ (lz - nx) + \omega_z\ (mx - ly). \qquad [135d]$$

If the equation of the surface of the boundary is given as

$$f\ (x, y, z) = 0, \qquad [135e]$$

the direction cosines of its normal at any point x, y, z can be found as

$$l = k \frac{\partial f}{\partial x}, \quad m = k \frac{\partial f}{\partial y}, \quad n = k \frac{\partial f}{\partial z}, \qquad [135f,g,h]$$

where

$$k = \left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2 \right]^{-1/2} \qquad [135i]$$

in order to make $l^2 + m^2 + n^2 = 1$. The sign of k must be determined by inspection.

For, if the point is displaced over the given surface through an elementary distance ds, whose components are dx, dy, dz, in the direction of a tangent whose direction cosines are l', m', n', from Equation [135e]

$$df = dx \frac{\partial f}{\partial x} + dy \frac{\partial f}{\partial y} + dz \frac{\partial f}{\partial z} = 0.$$

Substituting here $dx = l'ds$, $dy = m'ds$, $dz = n'ds$ and multiplying through by k/ds,

$$l'k \frac{\partial f}{\partial x} + m'k \frac{\partial f}{\partial y} + n'k \frac{\partial f}{\partial z} = 0.$$

Now a line can certainly be drawn through (x, y, z) whose direction cosines are l, m, n as defined by Equations [135f,g,h]. Then, from the last equation, $l'l + m'm + n'n = 0$, so that the line thus drawn is perpendicular to the tangent whose direction is (l', m', n'). Since the latter may be any tangent to the surface at (x, y, z), the line (l, m, n) must be the normal to the surface.

136. GENERAL FORMULAS FOR ORTHOGONAL CURVILINEAR COORDINATES

It is convenient at this point to generalize certain ideas and formulas so that they may be used with any type of orthogonal coordinates.

A coordinate system may be regarded as set up by means of three families of coordinate surfaces. The surfaces of any one family do not cut each other, and are numbered with the values of one of the coordinates. Usually the three surfaces that intersect at a given point meet there orthogonally.

For example, for Cartesian coordinates the surfaces consist of three sets of parallel planes. For the polar coordinates defined in Section 7, the surfaces are concentric spheres for r, cones for θ and planes through the axis for ω. For cylindrical coordinates as defined in Section 7 the surfaces are planes perpendicular to the axis for x, cylinders for ϖ, and planes through the axis for ω.

At any point there are three coordinate directions, in each of which one coordinate increases while the other two remain constant; see Figure 225. These directions are tangent to the curves of intersection of the three coordinate surfaces through the point. If the coordinates are orthogonal, the three coordinate directions at a given point are mutually perpendicular, and they are also perpendicular to the corresponding coordinate surfaces.

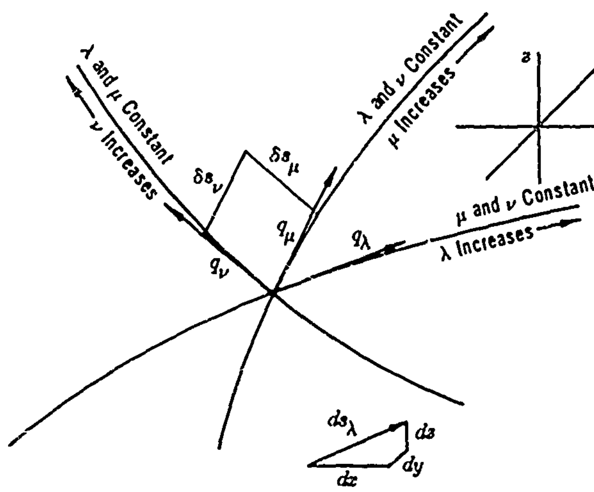


Figure 225 – Diagram illustrating coordinate directions. See Section 136.

When the coordinate λ of a point is given an elementary increase $\delta\lambda$ while the other two coordinates remain fixed, the variables x, y, and z receive certain elementary increments which can be written

$$\delta x_\lambda = \frac{\partial x}{\partial \lambda}\,\delta\lambda, \quad \delta y_\lambda = \frac{\partial y}{\partial \lambda}\,\delta\lambda, \quad \delta z_\lambda = \frac{\partial z}{\partial \lambda}\,\delta\lambda.$$

The total displacement of the point is then

$$\delta s_\lambda = (\delta x_\lambda^2 + \delta y_\lambda^2 + \delta z_\lambda^2)^{1/2}$$

and

$$\frac{\delta s_\lambda}{\delta\lambda} = \left[\left(\frac{\partial x}{\partial\lambda}\right)^2 + \left(\frac{\partial y}{\partial\lambda}\right)^2 + \left(\frac{\partial z}{\partial\lambda}\right)^2\right]^{1/2}. \qquad [136a]$$

Furthermore, let l, m, n be the direction cosines of the coordinate direction for λ, which is the direction of the displacement δs_λ. Then

$$\delta x_\lambda = l\delta s_\lambda,\ \delta y_\lambda = m\,\delta s_\lambda,\ \delta z_\lambda = n\delta s_\lambda.$$

Hence

$$l = \frac{\delta\lambda}{\delta s_\lambda}\ \frac{\partial x}{\partial\lambda},\ m = \frac{\delta\lambda}{\delta s_\lambda}\ \frac{\partial y}{\partial\lambda},\ n = \frac{\delta\lambda}{\delta s_\lambda}\ \frac{\partial z}{\partial\lambda}. \qquad [136b,c,d]$$

If the coordinates are orthogonal, l, m, n are also the direction cosines of the normal to the surface λ = constant. The ratio $\delta s_\lambda/\delta\lambda$ is easily calculated from the formulas connecting the coordinates with x, y, z.

For cartesian coordinates this ratio is unity. For spherical polar coordinates r, θ, ω as defined in Section 7, $\delta s_r/\delta r = 1$, $\delta s_\theta/\delta\theta = r$, $\delta s_\omega/\delta\omega = r\sin\theta$; for, increasing ω by $\delta\omega$, for example, displaces the point (r, θ, ω) through a distance $r\sin\theta\delta\omega$ along a circle of radius $r\sin\theta$ whose axis is the polar axis. For cylindrical coordinates x, ϖ, ω as defined in Section 7, $\delta s_x/\delta x = 1$, $\delta s_\varpi/\delta\varpi = 1$, $\delta s_\omega/\delta\omega = \varpi$, since the variation $\delta\omega$ produces a displacement $\varpi\delta\omega$ along a circle of radius ϖ.

In a flow having a velocity potential ϕ, the component of the velocity in the coordinate direction of any coordinate λ can be written, from Equation [6f],

$$q_\lambda = -\frac{\partial\phi}{\partial s} = -\frac{\delta\lambda}{\delta s_\lambda}\ \frac{\partial\phi}{\partial\lambda}, \qquad [136e]$$

since in this direction $ds = \delta s_\lambda$, $d\phi = \delta\lambda\ \partial\phi/\partial\lambda$;
see Figure 225. If the three coordinates λ, μ, ν are orthogonal, the magnitude of the velocity q is given by

$$q^2 = q_\lambda^2 + q_\mu^2 + q_\nu^2.$$

A general form of the Laplace equation may be obtained by expressing the continuity equation for an incompressible fluid in terms of the orthogonal curvilinear coordinates λ, μ, ν. Consider the element of volume bounded by the six surfaces that are defined by the following equations:

$$\begin{array}{lll} \lambda = \lambda_1 & \mu = \mu_1 & \nu = \nu_1 \\ \lambda = \lambda_1 + \delta\lambda & \mu = \mu_1 + \delta\mu & \nu = \nu_1 + \delta\nu \end{array}$$

where λ_1, μ_1, ν_1 refer to any given point in space. If $\delta\lambda$, $\delta\mu$, $\delta\nu$ are small, the element is sensibly rectangular in shape, as illustrated in Figure 226. Since the fluid is assumed to be incompressible, as much fluid must enter this element as leaves it.

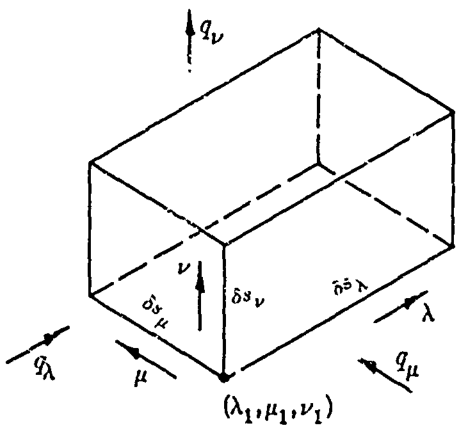


Figure 226 – Illustrating the equation of continuity in terms of the orthogonal curvilinear coordinates λ, μ, ν.

Consider first the pair of faces approximately perpendicular to the coordinate direction for λ. The face at which $\lambda = \lambda_1$ has sides of length δs_μ, δs_ν and an area $\delta s_\mu \ \delta s_\nu$. Fluid is entering the element across this face at a rate $q_\lambda \ \delta s_\mu \ \delta s_\nu$. The rate at which it is leaving the element across the opposite face, on which $\lambda = \lambda_1 + \delta\lambda$, can be written

$$q_\lambda \ \delta s_\mu \ \delta s_\nu + \delta\lambda \ \frac{\partial}{\partial\lambda} \ (q_\lambda \ \delta s_\mu \ \delta s_\nu).$$

The difference between this expression and the last, or

$$\delta\lambda \ \frac{\partial}{\partial\lambda} \ (q_\lambda \ \delta s_\mu \ \delta s_\nu),$$

is the net rate at which fluid is leaving the element by passing across this pair of faces.

Treating the other two pairs of faces in a similar way, and adding the three expressions thus found to obtain the total rate of outflow, which must be zero, it is found that

$$\delta\lambda \ \frac{\partial}{\partial\lambda} \ (q_\lambda \ \delta s_\mu \ \delta s_\nu) + \delta\mu \ \frac{\partial}{\partial\mu} \ (q_\mu \ \delta s_\nu \ \delta s_\lambda) + \delta\nu \ \frac{\partial}{\partial\nu} \ (q_\nu \ \delta s_\lambda \ \delta s_\mu) = 0.$$

Dividing by $\delta\lambda \, \delta\mu \, \delta\nu$ and noting that $\delta\lambda$, $\delta\mu$, $\delta\nu$ are constants,

$$\frac{\partial}{\partial\lambda}\left(\frac{\delta s_\mu}{\delta\mu} \ \frac{\delta s_\nu}{\delta\nu} \ q_\lambda\right) + \frac{\partial}{\partial\mu}\left(\frac{\delta s_\nu}{\delta\nu} \ \frac{\delta s_\lambda}{\delta\lambda} \ q_\mu\right) + \frac{\partial}{\partial\nu}\left(\frac{\delta s_\lambda}{\delta\lambda} \ \frac{\delta s_\mu}{\delta\mu} \ q_\nu\right) = 0. \quad [136f]$$

This is the equation of continuity for an incompressible fluid expressed in terms of any orthogonal coordinates.

If the motion is irrotational, substitution in the last equation from such equations as [136e] gives the Laplace equation for the potential ϕ:

$$\frac{\partial}{\partial\lambda}\left(\frac{\delta s_\mu}{\delta\mu}\frac{\delta s_\nu}{\delta\nu}\frac{\delta\lambda}{\delta s_\lambda}\frac{\partial\phi}{\partial\lambda}\right)+\frac{\partial}{\partial\mu}\left(\frac{\delta s_\nu}{\delta\nu}\frac{\delta s_\lambda}{\delta\lambda}\frac{\delta\mu}{\delta s_\mu}\frac{\partial\phi}{\partial\mu}\right)+\frac{\partial}{\partial\nu}\left(\frac{\delta s_\lambda}{\delta\lambda}\frac{\delta s_\mu}{\delta\mu}\frac{\delta\nu}{\delta s_\nu}\frac{\partial\phi}{\partial\nu}\right)=0. \qquad [136g]$$

In some problems the mass of fluid under consideration is actually bounded by a coordinate surface. For example, let λ be constant over the boundary. Then, provided the fluid at infinity is at rest, formula [17c] for the kinetic energy T of the fluid can be written

$$T=\frac{1}{2}\rho\left|\iint\frac{\delta s_\mu}{\delta\mu}\frac{\delta s_\nu}{\delta\nu}\phi\, q_\lambda\, d\mu\, d\nu\right|=\frac{1}{2}\rho\left|\iint\frac{\delta\lambda}{\delta s_\lambda}\frac{\delta s_\mu}{\delta\mu}\frac{\delta s_\nu}{\delta\nu}\phi\frac{\partial\phi}{\partial\lambda}\,d\mu\, d\nu\right| \qquad [136h]$$

where ρ is the density of the fluid and the surface integral extends over the entire finite boundary. For, the element of area on the surface dS can be taken in the form of an elementary rectangle with sides drawn in coordinate directions, so that along two opposite sides μ changes by $d\mu = \delta\mu$, and along the other two ν changes by $d\nu = \delta\nu$; see Figure 225. Thus dS can be replaced by the area of this rectangle or

$$\delta s_\mu\, \delta s_\nu = (\delta s_\mu/\delta\mu)\,(\delta s_\nu/\delta\nu)\; d\mu d\nu.$$

The normal component of the velocity is $q_n = \pm\, q_\lambda$ where q_λ is given by Equation [136e]. The sign in this latter equation is necessarily the same over the coordinate surface; and the absolute value of the integral is taken because T is necessarily positive.

For *axisymmetric* flow, the angle ω around the axis of symmetry is usually employed as one orthogonal coordinate: the other two, say λ and μ, then function as two-dimensional coordinates on any plane drawn through the axis. Any orthogonal coordinates may be used for λ and μ. The following relations between the λ and μ components of the velocity and the axisymmetric stream function ψ may be noted:

$$q_\lambda = \mp\frac{1}{\varpi}\frac{\delta\mu}{\delta s_\mu}\frac{\partial\psi}{\partial\mu}, \quad q_\mu = \pm\frac{1}{\varpi}\frac{\delta\lambda}{\delta s_\lambda}\frac{\partial\psi}{\partial\lambda}, \qquad [136i, j]$$

where ϖ denotes distance from the axis, which may be expressed in terms of λ and μ. The proper sign to use in these equations is easily chosen in a given case, or the following rule may be used: at a given point, the upper sign is to be taken in both equations when the coordinate direction for λ is carried into that for μ by a rotation of 90 deg in the direction from the assumed positive end of the axis of symmetry toward the point as in Figure 227: otherwise,

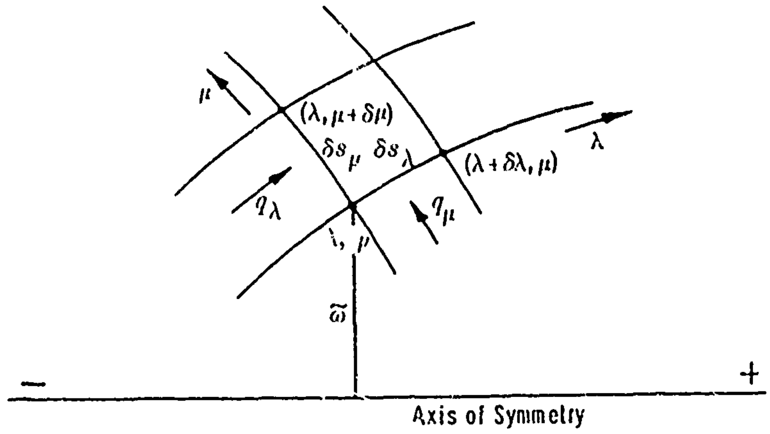


Figure 227 – Relations in axisymmetric flow.
See Section 136.

the lower signs are to be taken. The equations are easily obtained from the definition of ψ, as given in Section 16. For example, the flow between two circles drawn with the axis of symmetry as axis and through two points (λ, μ) and $(\lambda + \delta\lambda, \mu)$ is $2\pi\delta\psi = \pm 2\pi\varpi\delta s_\lambda q_\mu$; insertion of $\delta\psi = \delta\lambda\, \partial\psi/\partial\lambda$ and division by $2\pi\varpi\delta s_\lambda$ gives Equation [136j].

If a velocity potential ϕ also exists, it follows by comparison of Equations [136i,j] with Equation [136e] and its analog for q_μ that

$$\frac{\delta\lambda}{\delta s_\lambda}\frac{\partial\phi}{\partial\lambda} = \pm\frac{1}{\varpi}\frac{\delta\mu}{\delta s_\mu}\frac{\partial\psi}{\partial\mu}, \quad \frac{\delta\mu}{\delta s_\mu}\frac{\partial\phi}{\partial\mu} = \mp\frac{1}{\varpi}\frac{\delta\lambda}{\delta s_\lambda}\frac{\partial\psi}{\partial\lambda}. \qquad [136k,l]$$

The signs are explained under Equations [136i, j]. Because of the symmetry, the third term in the Laplace Equation [136g], in which now $\nu = \omega$, disappears. A corresponding equation for ψ is obtained by substituting for $\partial\phi/\partial\lambda$ and $\partial\phi/\partial\mu$ from Equations [136k, l] in the identity $\partial^2\phi/\partial\mu\partial\lambda = \partial^2\phi/\partial\lambda\partial\mu$;

$$\frac{\partial}{\partial\lambda}\left(\frac{1}{\varpi}\frac{\delta s_\mu}{\delta\mu}\frac{\delta\lambda}{\delta s_\lambda}\frac{\partial\psi}{\partial\lambda}\right) + \frac{\partial}{\partial\mu}\left(\frac{1}{\varpi}\frac{\delta s_\lambda}{\delta\lambda}\frac{\delta\mu}{\delta s_\mu}\frac{\partial\psi}{\partial\mu}\right) = 0. \qquad [136m]$$

137. OVARY ELLIPSOIDS (OR PROLATE SPHEROIDS)

Problems involving an ellipsoid are most easily handled in terms of some form of ellipsoidal coordinates. Special types are used when two axes are equal.

For an ovary ellipsoid, or prolate spheroid, the prolate-spheroidal coordinates ζ, μ, ω are most conveniently defined inversely, thus:

$$x = k\mu\zeta,\ y = \varpi \cos \omega,\ z = \varpi \sin \omega, \qquad [137a, b, c]$$

$$\varpi = k(\zeta^2 - 1)^{1/2} (1 - \mu^2)^{1/2}, \qquad [137d]$$

where k is an arbitrary positive constant and positive values of the radicals are intended. Thus

$$r^2 = x^2 + y^2 + z^2 = x^2 + \varpi^2 = k^2 (\zeta^2 + \mu^2 - 1). \qquad [137e]$$

Here ω is an angle representing position about the x-axis; ζ, μ, and ω are dimensionless, whereas k represents a fundamental length. The coordinate surfaces for ζ and μ are confocal ellipsoids and hyperboloids of revolution, with foci on the x-axis at $x = \pm k$; their equations, obtained by eliminating either μ or ζ, are

$$\frac{x^2}{k^2 \zeta^2} + \frac{\varpi^2}{k^2 (\zeta^2 - 1)} = 1,\ \frac{x^2}{k^2 \mu^2} - \frac{\varpi^2}{k^2 (1 - \mu^2)} = 1. \qquad [137f, g]$$

The traces of the coordinate surfaces on any plane through the x-axis are confocal ellipses and hyperbolas; it was seen in Section 61 that such curves cut each other orthogonally. It will be simplest to treat the two halves of such a plane as separate planes, distinguished by complementary values of ω. On each half-plane either x and ϖ or ζ and μ then serve as single-valued coordinates and $\varpi \geqq 0$. Convenient ranges of values for ζ and μ, as indicated in Figure 228, are: $1 \leqq \zeta$, $-1 \leqq \mu \leqq 1$.

The coordinates ζ, μ are simply the elliptic coordinates of Section 82 in disguise: $\zeta = \cosh \xi$, $\mu = \cos \eta$, and here $k = c$. Formulas for ζ and μ in terms of x and y can be written down at once from Equations [82e, f].

The semiaxes of any ζ ellipsoid and its ellipticity are

$$a' = k\zeta,\ b' = k (\zeta^2 - 1)^{1/2},\ e' = 1/\zeta. \qquad [137h, i, j]$$

In terms of these, $x = a'\mu$, $\varpi = b'\sqrt{1 - \mu^2}$. Also, $k = a'e'$. On the x-axis, for $x \geqq k$, $\mu = 1$, $x = k\zeta$: for $x \leqq -k$, $\mu = -1$, $x = -k\zeta$. For $|x| \leqq k$, $\zeta = 1$, $a' = k$ and $x = k\mu$. On the ϖ-axis, $\mu = 0$ and $\varpi = k\sqrt{\zeta^2 - 1}$.

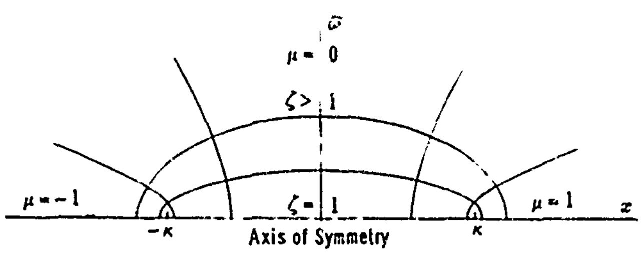


Figure 228 – Choice of signs for prolate-spheroidal coordinates. See Section 137.

Toward infinity, $\zeta \to \infty$ and, approximately, $\varpi = k\zeta\,(1 - \mu^2)^{1/2}$ and $r = (x^2 + \varpi^2)^{1/2} = k\zeta$, so that $\zeta = r/k$, $\mu = x/k\zeta = x/r = \cos\theta$ in terms of the polar angle θ.

The elements of distance in the coordinate directions, calculated from Equation [136a], are

$$\delta s_\zeta = k\left(\frac{\zeta^2 - \mu^2}{\zeta^2 - 1}\right)^{1/2} \delta\zeta,\quad \delta s_\mu = k\left(\frac{\zeta^2 - \mu^2}{1 - \mu^2}\right)^{1/2} \delta\mu, \qquad [137k, l]$$

$$\delta s_\omega = \varpi\delta\omega = k\,(\zeta^2 - 1)^{1/2}\,(1 - \mu^2)^{1/2}\,\delta\omega. \qquad [137m]$$

The coordinate direction for ω is perpendicular to the plane through the x-axis; that for ζ is perpendicularly outward across the ellipsoids, that for μ is tangential to them and from $\mu < 0$ or $x < 0$ around toward $\mu > 0$ or $x > 0$. These two directions make angles θ_ζ, θ_μ with the positive x-axis such that $0 \leqq \theta_\zeta \leqq \pi$, $-\pi/2 \leqq \theta_\mu \leqq \pi/2$, and, from Equations [136b, c, d],

$$\cos\theta_\zeta = -\sin\theta_\mu = \mu\left(\frac{\zeta^2 - 1}{\zeta^2 - \mu^2}\right)^{1/2}. \qquad [137n]$$

The components of velocity in the coordinate directions are, from Equation [136e],

$$q_\zeta = -\frac{\delta\zeta}{\delta s_\zeta}\frac{\partial\phi}{\partial\zeta},\quad q_\mu = -\frac{\delta\mu}{\delta s_\mu}\frac{\partial\phi}{\partial\mu},\quad q_\omega = -\frac{\delta\omega}{\delta s_\omega}\frac{\partial\phi}{\partial\omega}. \qquad [137o, p, q]$$

Laplace's equation for the potential ϕ becomes, from Equation [136g],

$$\frac{\partial}{\partial\zeta}\left[(\zeta^2 - 1)\ \frac{\partial\phi}{\partial\zeta}\right] + \frac{\partial}{\partial\mu}\left[(1 - \mu^2)\ \frac{\partial\phi}{\partial\mu}\right] + \frac{\zeta^2 - \mu^2}{(1 - \mu^2)(\zeta^2 - 1)}\ \frac{\partial^2\phi}{\partial\omega^2} = 0; \quad [137r]$$

and the antisymmetric stream function ψ, defined on the basis of a positive axis drawn toward $\mu = 1$, according to Equations [136k, l], is related to ϕ as follows:

$$\frac{\partial\psi}{\partial\zeta} = k\,(1 - \mu^2)\ \frac{\partial\phi}{\partial\mu}\ ,\quad \frac{\partial\psi}{\partial\mu} = -\,k\,(\zeta^2 - 1)\ \frac{\partial\phi}{\partial\zeta}\,. \qquad [137s, t]$$

Suppose now, that a solid ellipsoid of revolution is given with a surface defined by

$$\frac{x^2}{a^2} + \frac{y^2 + z^2}{b^2} = 1,\ a > b. \qquad [137u]$$

Then its ellipticity is $e = (a^2 - b^2)^{1/2}/a$, so that $b = a\sqrt{1 - e^2}$; and for this ellipsoid $a' = a,\ b' = b,\ \zeta = \zeta_0 = a/k = 1/e$. Thus $k = ea = \sqrt{a^2 - b^2}$, and on this ellipsoid $x = a\mu,\ \varpi = b\,(1 - \mu^2)^{1/2}$.

Five cases of the flow around the solid ellipsoid will be treated. In each case ϕ as stated may be verified to satisfy Equation [137r], and ψ, if it exists, to satisfy Equations [137s, t]. The general case can be constructed by superposing flows of two or more of these five types.

Case 1. Translation of a Prolate Spheroid Parallel to its Axis of Symmetry at velocity U, toward $\mu = 1$, with $q = 0$ in the fluid at infinity; see Figure 229.

$$\phi = g_1\,k\,U\,\mu\left(\frac{1}{2}\ \zeta\ln\frac{\zeta + 1}{\zeta - 1} - 1\right), \qquad [137v]$$

$$\psi = -\,\frac{1}{2}\ g_1\,k^2\,U\,(\zeta^2 - 1)\,(1 - \mu^2)\left(\frac{\zeta}{\zeta^2 - 1} - \frac{1}{2}\ \ln\frac{\zeta + 1}{\zeta - 1}\right), \qquad [137w]$$

$$g_1 = \left(\frac{\zeta_0}{\zeta_0^2 - 1} - \frac{1}{2}\ \ln\frac{\zeta_0 + 1}{\zeta_0 - 1}\right)^{-1} = \left(\frac{e}{1 - e^2} - \frac{1}{2}\ \ln\frac{1 + e}{1 - e}\right)^{-1}. \qquad [137x]$$

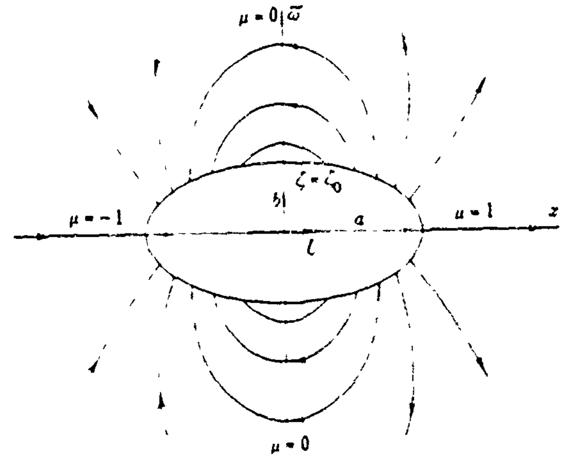


Figure 229 – Streamlines due to translation of a prolate spheroid in the direction of its axis. The foci are shown by dots. See Section 137, Case 1.

Now, for any number ξ,

$$\ln \frac{\xi+1}{\xi-1} = \ln\left(1+\frac{1}{\xi}\right) - \ln\left(1-\frac{1}{\xi}\right), \quad \frac{\xi}{\xi^2-1} = \frac{1}{\xi}\left(1-\frac{1}{\xi^2}\right)^{-1},$$

hence, expanding, for $|\xi| > 1$,

$$\ln \frac{\xi+1}{\xi-1} = \frac{1}{\xi} - \frac{1}{2\xi^2} + \frac{1}{3\xi^3} - \frac{1}{4\xi^4} + \frac{1}{5\xi^5} \cdots\cdots - \left(-\frac{1}{\xi} - \frac{1}{2\xi^2} - \frac{1}{3\xi^3} - \frac{1}{4\xi^4} - \frac{1}{5\xi^5} \cdots\cdots\right) = \frac{2}{\xi} + \frac{2}{3\xi^3} + \frac{2}{5\xi^5} \cdots\cdots \qquad [137y]$$

$$\frac{\xi}{\xi^2-1} = \frac{1}{\xi} + \frac{1}{\xi^3} + \frac{1}{\xi^5} \cdots\cdots. \qquad [137z]$$

Hence, if $e \to 0$ and thus $\zeta_0 \to \infty$, $e^3 g_1 = g_1/\zeta_0^3 = (2/3 + \cdots\cdots)^{-1} \to 3/2$. Again, as $e \to 1$ and $\zeta_0 \to 1$, both $\ln(1+e)$ and $\ln(1-e)$ become numerically negligible in comparison with $e/(1-e^2)$; hence $g_1 \to 0$, $g_1/(1-e^2) \to 1$.

Toward infinity $\zeta \to \infty$ and, from Equation [137v], since $\zeta \to r/k$ and $k = ea$,

$$\phi = g_1 k U \mu\left(\frac{1}{3\zeta^2} + \cdots\cdots\right) = \frac{1}{3} e^3 g_1 a^3 U \frac{\cos\theta}{r^2} \quad \text{approximately.}$$

Thus the flow becomes that of a dipole of moment $e^3 g_1 a^3 U/3$ located at the center of the ellipsoid. If e is small, the coefficient $e^3 g_1/3$ becomes 1/2, as for a sphere; see Section 127.

The velocity components are $q_\omega = 0$ and

$$q_\zeta = g_1 U \mu \left(\frac{\zeta^2 - 1}{\zeta^2 - \mu^2}\right)^{1/2} \left(\frac{\zeta}{\zeta^2 - 1} - \frac{1}{2} \ln \frac{\zeta + 1}{\zeta - 1}\right), \qquad [137a']$$

$$q_\mu = -g_1 U \left(\frac{1 - \mu^2}{\zeta^2 - \mu^2}\right)^{1/2} \left(\frac{1}{2} \zeta \ln \frac{\zeta + 1}{\zeta - 1} - 1\right). \qquad [137b']$$

On the x-axis, $\mu = \pm 1$, $|x| = k\zeta$, $q = |u|$ and

$$u = \pm q_\zeta = g_1 U \left(\frac{k|x|}{x^2 - k^2} - \frac{1}{2} \ln \frac{|x| + k}{|x| - k}\right); \qquad [137c']$$

on the equatorial or yz-plane $\mu = 0$, $\varpi = k(\zeta^2 - 1)^{1/2}$, $q = |u|$ and

$$u = q_\mu = -g_1 U \left(\frac{1}{2} \ln \frac{\sqrt{\varpi^2 + k^2} + k}{\sqrt{\varpi^2 + k^2} - k} - \frac{k}{\sqrt{\varpi^2 + k^2}}\right) \qquad [137d']$$

on the equatorial circumference of the ellipsoid itself $\zeta = \zeta_0 = 1/e$ and

$$u = q_\mu = -g_1 U \left(\frac{1}{2} \ln \frac{1 + e}{1 - e} - e\right). \qquad [137e']$$

A few streamlines for equidistant values of ψ are shown in Figure 229. Here $a/b = 2$, $e = 0.866$, $g_1 = 0.466$.

The kinetic energy of the fluid, as found by substituting ζ, μ, ω for λ, μ, ν in Equation [136h], using Equations [137k, l, m] and [137v], $k = ae$, $b^2 = a^2(1 - e^2)$, $\int_{-1}^{1} \mu^2 \, d\mu = 2/3$, $\int_0^{2\pi} d\omega = 2\pi$, and setting $\zeta = \zeta_0 = 1/e$, is

$$T = \frac{2}{3} \pi \rho a b^2 U^2 \left(\frac{e^3 g_1}{1 - e^2} - 1\right). \qquad [137f']$$

Case 2. Flow Past a Prolate Spheroid Parallel to the Axis of Symmetry. Let the fluid at infinity flow at velocity U toward $\mu = -1$. Adding, from Equations [119a, b], Ux to ϕ and $U\varpi^2/2$ to ψ as given by Equations [137v, w], to represent the superposed uniform flow,

$$\phi = kU\mu \left[\zeta + g_1 \left(\frac{1}{2} \zeta \ln \frac{\zeta + 1}{\zeta - 1} - 1\right)\right], \qquad [137g']$$

$$\psi = \frac{1}{2} k^2 U (\zeta^2 - 1)(1 - \mu^2) \left[1 - g_1 \left(\frac{\zeta}{\zeta^2 - 1} - \frac{1}{2} \ln \frac{\zeta + 1}{\zeta - 1} \right) \right]. \qquad [137h']$$

Hence $q_\omega = 0$, and, if a prime denotes values as given by Equations [137a', b'].

$$q_\zeta = q_\zeta' - U\mu \left(\frac{\zeta^2 - 1}{\zeta^2 - \mu^2} \right)^{1/2}, \quad q_\mu = q_\mu' - U\zeta \left(\frac{1 - \mu^2}{\zeta^2 - \mu^2} \right)^{1/2}. \qquad [137i', j'].$$

on the x-axis and in the yz-plane $q = |u|$ and $u = u' - U$.

On the ellipsoid itself, where $\zeta = \zeta_0 = 1/e$, $\psi = 0$, so that this is a stream surface. Also, $q_\zeta = 0$, $q = |q_\mu|$ and

$$q_\mu = - \frac{g_1 U}{\zeta_0^2 - 1} \left(\frac{1 - \mu^2}{\zeta_0^2 - \mu^2} \right)^{1/2} = - \frac{e^3 g_1 U}{1 - e^2} \left(\frac{a^2 - x^2}{a^2 - e^2 x^2} \right)^{1/2}, \qquad [137k']$$

since $x = a\mu$. As $\zeta_0 \to \infty$, $e \to 0$, $e^3 g_1 \to 3/2$, and $q_\mu \to -(3/2)\, U \sqrt{a^2 - x^2}/a$, as for a sphere. As $\zeta \to 1$, $e \to 1$, $g_1 \to 1 - e^2$ and $q_\mu \to -U$; the ellipse has then become a cylinder.

Some of the streamlines, for $e = 0.866$, $g_1 = 0.466$, are shown in Figure 230; they are selected to be equally spaced at infinity. The excess of pressure above that in the stream,

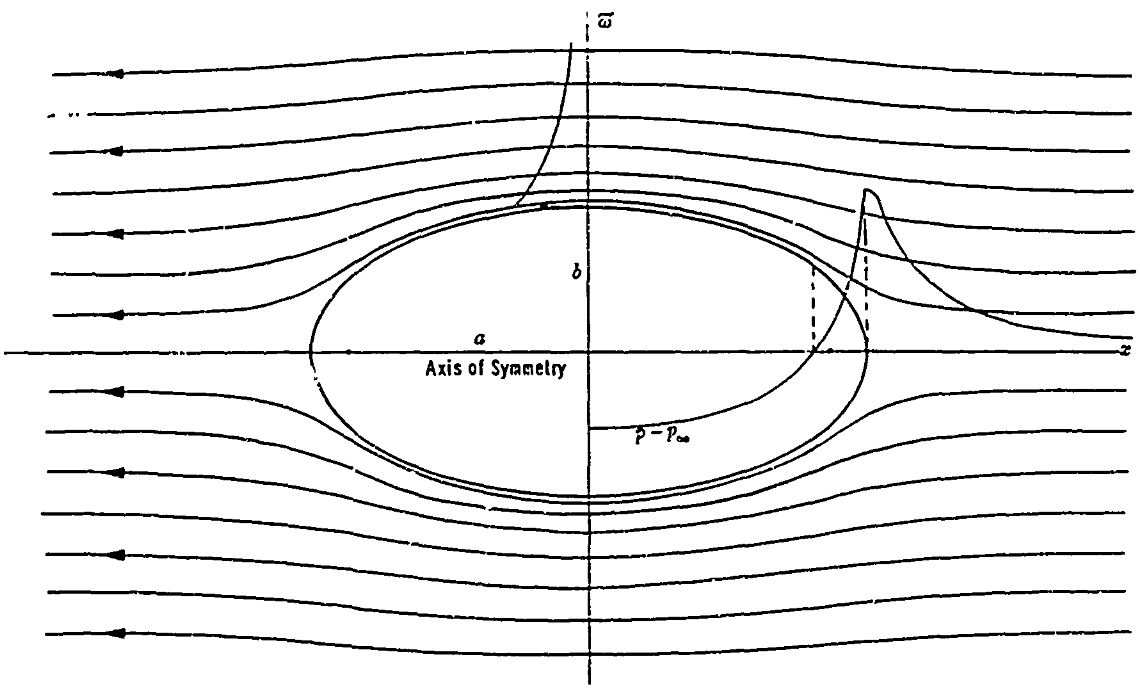


Figure 230 – Streamlines for flow past a prolate spheroid in the direction of its major axis. The pressure is shown along the axis, over the spheroid and outward along a minor axis. The foci are shown by dots. See Section 137, Case 2.

for steady motion, is also plotted as $p - p_\infty$, on an arbitrary scale. It is shown along the x-axis, then over the ellipsoid, where $p = p_\infty$ at $x = 0.807a$, and along the y-axis, where negative values are plotted horizontally toward the left.

Case 3. Translation of a Prolate Spheroid Perpendicularly to Its Axis of Symmetry at velocity V toward positive y or $\omega = 0$; see Figure 231.

$$\phi = h_1 k V (\zeta^2 - 1)^{1/2} (1 - \mu^2)^{1/2} \left[\frac{\zeta}{\zeta^2 - 1} - \frac{1}{2} \ln \frac{\zeta + 1}{\zeta - 1} \right] \cos \omega, \qquad [137l']$$

$$h_1 = \left(\frac{1}{2} \ln \frac{\zeta_0 + 1}{\zeta_0 - 1} - \frac{\zeta_0^2 - 2}{\zeta_0(\zeta_0^2 - 1)} \right)^{-1} = \left(\frac{1}{2} \ln \frac{1 + e}{1 - e} - \frac{e(1 - 2e^2)}{1 - e^2} \right)^{-1}. \qquad [137m']$$

As $e \to 1$, and $\zeta_0 \to 0$, the logarithmic term becomes unimportant and $h_1 \to 0$, $h_1/(1 - e^2) \to 1$. As $e \to 0$ and $\zeta_0 \to \infty$, $e^3 h_1 \to 3/4$, as appears from Equation [137y] and the expansion,

$$\frac{\zeta_0^2 - 2}{\zeta_0(\zeta_0^2 - 1)} = \frac{1}{\zeta_0} \left(1 - \frac{2}{\zeta_0^2} \right) \left(1 - \frac{1}{\zeta_0^2} \right)^{-1} = \frac{1}{\zeta_0} - \frac{1}{\zeta_0^3} \cdots \qquad [137n']$$

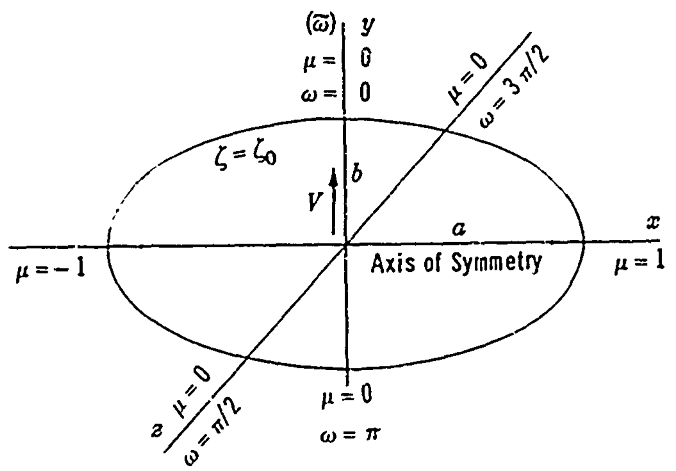


Figure 231 – Diagram for translation of a prolate spheroid in the direction of a minor axis. The foci are shown by dots. See Section 137, Case 3.

Toward infinity $\zeta \to r/k \to \infty$, $(\zeta^2 - 1)^{1/2} \to \zeta$ and, from Equations [137y, z], approximately,

$$\phi = \frac{2}{3} h_1 k V (1 - \mu^2)^{1/2} \frac{\cos \omega}{\zeta^2} = \frac{2}{3} e^3 h_1 a^3 V \frac{y}{r^3}, \qquad [137o']$$

since $k = ae$. The flow is again that of a dipole, but this time with its axis in the y direction; and for small values of e agreement with the result for a sphere is obtained, since then $2 e^3 h_1/3 \to 1/2$.

The velocity components in the coordinate directions are, from Equations [137o, p, q] and Equations [137k, l, m],

$$q_\zeta = h_1 V \left(\frac{1 - \mu^2}{\zeta^2 - \mu^2} \right)^{1/2} \left(\frac{1}{2} \zeta \ln \frac{\zeta + 1}{\zeta - 1} - \frac{\zeta^2 - 2}{\zeta^2 - 1} \right) \cos \omega, \qquad [137p']$$

$$q_\mu = h_1 V \mu \left(\frac{\zeta^2 - 1}{\zeta^2 - \mu^2} \right)^{1/2} \left(\frac{\zeta}{\zeta^2 - 1} - \frac{1}{2} \ln \frac{\zeta + 1}{\zeta - 1} \right) \cos \omega, \qquad [137q']$$

$$q_\omega = h_1 V \left(\frac{\zeta}{\zeta^2 - 1} - \frac{1}{2} \ln \frac{\zeta + 1}{\zeta - 1} \right) \sin \omega. \qquad [137r']$$

On the axes the velocity is in the direction of $\pm V$ and $q = |v|$. On the y-axis, $\cos \omega = \pm 1$, $\mu = 0$, $y = \pm k(\zeta^2 - 1)^{1/2}$ and

$$v = \pm q_\zeta = h_1 V \left[\ln \left(\frac{k}{|y|} + \sqrt{1 + \frac{k^2}{y^2}} \right) - \frac{k}{\sqrt{y^2 + k^2}} \left(1 - \frac{k^2}{y^2} \right) \right]. \qquad [137s']$$

On the x-axis $\mu = \pm 1$, $x = \pm k \zeta$ and

$$v = - h_1 V \left(\frac{k\,|x|}{x^2 - k^2} - \frac{1}{2} \ln \frac{|x| + k}{|x| - k} \right),$$

which represents the limit of $\mp q_\mu$ as $|\mu| \to 1$ while $\omega = 0$. On the z-axis, $\mu = 0$, $\sin \omega = \pm 1$, $z = \pm k (\zeta^2 - 1)^{1/2}$ and

$$v = \mp q_\omega = - h_1 V \left(\frac{k}{z^2} \sqrt{z^2 + k^2} - \frac{1}{2} \ln \frac{\sqrt{z^2 + k^2} + k}{\sqrt{z^2 + k^2} - k} \right). \qquad [137t']$$

Over the surface of the ellipsoid itself, where $\zeta = \zeta_0$ and is constant, the transverse component q_ω varies relatively in the same manner as over a sphere. Furthermore, around the circumference in the zx-plane, where $\sin\omega = \pm 1$, q is constant, since

$$q = |q_\omega| = h_1 V\left(\frac{\zeta_0}{\zeta_0^2 - 1} - \frac{1}{2}\ln\frac{\zeta_0 + 1}{\zeta_0 - 1}\right) = h_1 V\left(\frac{e}{1 - e^2} - \frac{1}{2}\ln\frac{1 + e}{1 - e}\right). \quad [137u']$$

The kinetic energy of the fluid, found from Equation [136h] in analogy with Equation [137f'] but with use of the integrals $\int_{-1}^{1}(1 - \mu^2)\,d\mu = 4/3$ and $\int_0^{2\pi}\cos^2\omega\,d\omega = \pi$, is

$$T = \frac{2}{3}\pi\rho h_1 ab^2 V^2\left(\frac{e}{1 - e^2} - \frac{1}{2}\ln\frac{1 + e}{1 - e}\right). \quad [137v']$$

Case 4. Flow Past a Prolate Spheroid Perpendicularly to Its Axis of Symmetry. Let the fluid at infinity flow at velocity V toward negative y or $\mu = 0$, $\omega = \pi$. Adding Vy for the uniform stream,

$$\phi = kV(\zeta^2 - 1)^{1/2}(1 - \mu^2)^{1/2}\left[1 + h_1\left(\frac{\zeta}{\zeta^2 - 1} - \frac{1}{2}\ln\frac{\zeta + 1}{\zeta - 1}\right)\right]\cos\omega. \quad [137w']$$

If a prime denotes values given by Equations [137p', q', r']; from Equations [137o,p,q],

$$q_\zeta = q_\zeta' - V\zeta\left(\frac{1 - \mu^2}{\zeta^2 - \mu^2}\right)^{1/2}\cos\omega, \quad q_\mu = q_\mu' + V\mu\left(\frac{\zeta^2 - 1}{\zeta^2 - \mu^2}\right)^{1/2}\cos\omega. \quad [137x', y']$$

$$q_\omega = q_\omega' + V\sin\omega. \quad [137z']$$

Everywhere $v = v' - V$, and on all three axes $q = |v|$.

On the ellipsoid itself, where $\zeta = \zeta_0 = 1/e$, $q_\zeta = 0$ and

$$q_\mu = \frac{2e^3 h_1 V}{(1 - e^2)^{1/2}}\,\frac{\mu\cos\omega}{(1 - e^2\mu^2)^{1/2}}, \quad q_\omega = \frac{2e^3 h_1 V}{1 - e^2}\sin\omega, \quad [137a'', b'']$$

where $\mu = x/a$. The same remarks concerning q_ω apply here as in Case 3, except that here around the circumference of the ellipsoid in the zx-plane $q = |q_\omega|$ where $|q_\omega|$ is given by

Equation [137b''] with sin ω omitted. The coefficient $2\, e^3 h_1/(1 - e^2)$ becomes 2, as for a cylinder, as $e \to 1$, and 3/2, as for a sphere, as $e \to 0$.

Case 5. Rotation of a Prolate Spheroid about an Equatorial Axis. Let the spheroid represented by Equation [137u] rotate at angular velocity Ω about the y-axis, in fluid at rest at infinity.

At any instant, the velocity potential ϕ can be expressed in terms of ellipsoidal coordinates whose axes coincide with those of the ellipsoid, and the component of the fluid velocity that is normal to the surface of the ellipsoid at any point will then be given by Equation [137o]. The direction cosines of the normal to the surface, on the other hand, are, from Equations [136b, c, d],

$$l = \frac{\delta\zeta}{\delta s_\zeta}\frac{\partial x}{\partial\zeta}, \qquad m = \frac{\delta\zeta}{\delta s_\zeta}\frac{\partial y}{\partial\zeta}, \qquad n = \frac{\delta\zeta}{\delta s_\zeta}\frac{\partial z}{\partial\zeta}$$

Substitution of these values for q_n, l, m, n, and of $\omega_x = \omega_z = 0$, $\omega_y = \Omega$, in Equation [135d] gives as the boundary condition at the surface of the rotating ellipsoid

$$\frac{\partial\phi}{\partial\zeta} = -\Omega\left(z\frac{\partial x}{\partial\zeta} - x\frac{\partial z}{\partial\zeta}\right). \qquad [137c'']$$

From Equations [137a, c]

$$\frac{\partial x}{\partial\zeta} = k\mu, \frac{\partial z}{\partial\zeta} = k\zeta(\zeta^2 - 1)^{-1/2}(1 - \mu^2)^{1/2}\sin\omega.$$

The following potential will be found to satisfy both the boundary condition and the Laplace Equation [137r]:

$$\phi = A\mu(\zeta^2 - 1)^{1/2}(1 - \mu^2)^{1/2}\left(\frac{3}{2}\zeta\, ln\frac{\zeta + 1}{\zeta - 1} - 3 - \frac{1}{\zeta^2 - 1}\right)\sin\omega \qquad [137d'']$$

$$A = k^2\Omega\left[\frac{3}{2}(2\zeta_0^2 - 1)\, ln\frac{\zeta_0 + 1}{\zeta_0 - 1} - 6\zeta_0 + \frac{\zeta_0}{\zeta_0^2 - 1}\right]^{-1}. \qquad [137e'']$$

At large distances, using Equations [137y, z],

$$\phi = -\frac{2}{5}\frac{A\mu}{\zeta^3}(1-\mu^2)^{1/2}\sin\omega = -\frac{2A\,zx}{5k^2\zeta^5} = -\frac{2}{5}Ae^3a^3\frac{zx}{r^5}$$

approximately, since $k\zeta \approx r$ and $k \approx ea$. Thus the disturbance of the fluid extends effectively to only a short distance.

The velocity components are

$$q_\zeta = \frac{A\mu}{k}\left(\frac{1-\mu^2}{\zeta^2-\mu^2}\right)^{1/2}\left[6\zeta - \frac{\zeta}{\zeta^2-1} - \frac{3}{2}(2\zeta^2-1)\,ln\,\frac{\zeta+1}{\zeta-1}\right]\sin\omega, \qquad [137f'']$$

$$q_\mu = \frac{A}{k}(1-2\mu^2)\left(\frac{\zeta^2-1}{\zeta^2-\mu^2}\right)^{1/2}\left(3+\frac{1}{\zeta^2-1}-\frac{3}{2}\zeta\,ln\,\frac{\zeta+1}{\zeta-1}\right)\sin\omega, \qquad [137g'']$$

$$q_\omega = \frac{A\mu}{k}\left(3+\frac{1}{\zeta^2-1}-\frac{3}{2}\zeta\,ln\,\frac{\zeta+1}{\zeta-1}\right)\cos\omega. \qquad [137h'']$$

The pressure in this case can be found from Equation [11c] or Equation [11d]. The kinetic energy is given in Reference 1 and in the table following Section 147, Case 29(3).

In the last three cases there is no axis of symmetry, hence no stream function exists.

An extensive comparison of the theoretical formulas for the pressure with observation, resulting in general good agreement except in the wake, was reported by Jones.[234] (See Reference 1, Article 105, 106; Reference 2, Section 15.57; Zahm.[102, 174])

138. PLANETARY ELLIPSOIDS (OR OBLATE SPHEROIDS) AND CIRCULAR DISKS

For an ellipsoid of planetary form, or an oblate spheroid, the treatment of the last section requires only minor modifications.

For this case, oblate-spheroidal coordinates ζ, μ, ω are defined thus:

$$x = k\mu\zeta, \qquad y = \varpi\cos\omega, \qquad z = \varpi\sin\omega, \qquad [138a, b, c]$$

$$\varpi = k(\zeta^2+1)^{1/2}(1-\mu^2)^{1/2}. \qquad [138d]$$

The surfaces ζ = constant are again ellipsoids of revolution, but those for μ = constant are now hyperboloids of one sheet, with circular apertures lying in the yz-plane; the equations are

$$\frac{x^2}{k^2\zeta^2} + \frac{\varpi^2}{k^2(\zeta^2+1)} = 1, \qquad \frac{\varpi^2}{k^2(1-\mu^2)} - \frac{x^2}{k^2\mu^2} = 1. \qquad [138e, f]$$

In any plane through the x-axis the intercepts are orthogonal ellipses and hyperbolas with common foci lying on the focal ring defined by $x = 0$, $\varpi = k$. See Figure 232, on which again only a half-plane is shown. Assume $\zeta \geqq 0$. Then $-1 \leqq \mu \leqq 1$.

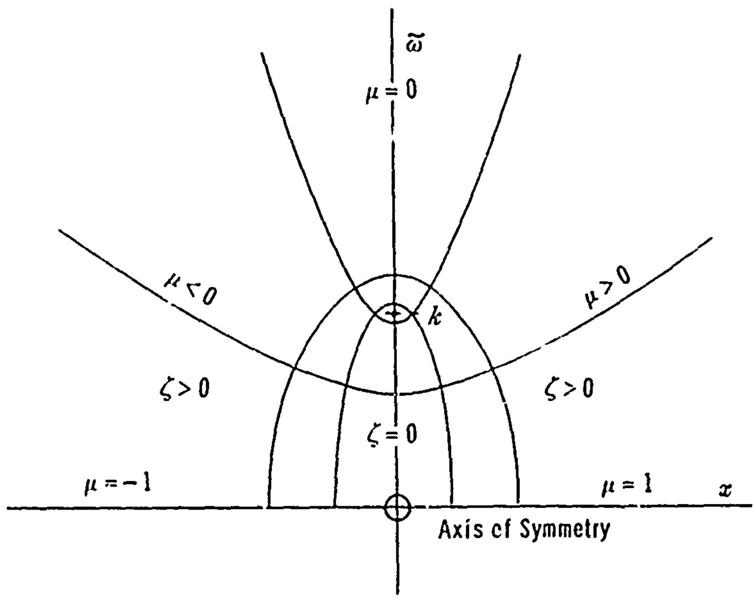


Figure 232 – Choice of signs for oblate-spheroidal coordinates.
See Section 138.

The relation with the elliptic coordinates of Section 82 is now: $k = c$, $\zeta = \sinh \xi$, $\mu = \sin \eta$; and x and y are replaced, respectively by ϖ and x. Formulas for ζ and μ in terms of x and y are easily written down from Equations [82e, f].

The polar and equatorial radii of any ellipsoid, in the x and ϖ directions, respectively, and the eccentricity of its meridian section are

$$a' = k\zeta, \quad c' = k(\zeta^2+1)^{1/2}, \quad e' = \sqrt{c'^2 - a'^2}/c' = 1/(\zeta^2+1)^{1/2}, \qquad [138g, h, i]$$

whence

$$k = e'c' = \sqrt{c'^2 - a'^2}, \quad \zeta = a'/k = a'/\sqrt{c'^2 - a'^2} = \sqrt{1 - e'^2}/e', \qquad [138j, k]$$

$$x = a'\mu, \quad \varpi = c'(1-\mu^2)^{1/2}. \qquad [138l, m]$$

The ellipsoid for $\zeta = 0$ is a circular disk of radius $\varpi = c' = k$ in the yz-plane, on which $\varpi = k\sqrt{1-\mu^2}$; the remainder of the yz-plane is the hyperboloid $\mu = 0$, on which $\varpi = k\sqrt{\zeta^2 + 1}$. The hyperboloid for $\mu = \pm 1$ is the entire x-axis, on which $x = \pm k\zeta$.

Toward infinity, $\zeta \to \infty$ and, approximately, $\varpi = k\zeta(1-\mu^2)^{1/2}$, $r = (x^2 + \varpi^2)^{1/2} = k\zeta$, so that $\zeta = r/k$, $\mu = x/k\zeta = x/r = \cos\theta$ in terms of the polar angle θ.

The coordinate elements of distance are

$$\delta s_\zeta = k\left(\frac{\zeta^2 + \mu^2}{\zeta^2 + 1}\right)^{1/2} \delta\zeta, \quad \delta s_\mu = k\left(\frac{\zeta^2 + \mu^2}{1 - \mu^2}\right)^{1/2} \delta\mu, \qquad [138n, o]$$

$$\delta s_\omega = \varpi\,\delta\omega = k(\zeta^2 + 1)^{1/2}(1-\mu^2)^{1/2}\,\delta\omega. \qquad [138p]$$

The coordinate directions for ζ and μ make angles θ_ζ, θ_μ with the positive x-axis which lie in the ranges $0 \leqq \theta_\zeta \leqq \pi$, $-\frac{\pi}{2} \leqq \theta_\mu \leqq \frac{\pi}{2}$ and are given by

$$\cos\theta_\zeta = -\sin\theta_\mu = \mu\left(\frac{\zeta^2 + 1}{\zeta^2 + \mu^2}\right)^{1/2}; \qquad [138q]$$

and the velocity components in the coordinate directions are given by Equations [137o, p, q] or

$$q_\zeta = -\frac{\delta\zeta}{\delta s_\zeta}\frac{\partial\phi}{\partial\zeta}, \quad q_\mu = -\frac{\delta\mu}{\delta s_\mu}\frac{\partial\phi}{\partial\mu}, \quad q_\omega = -\frac{\partial\omega}{\delta s_\omega}\frac{\partial\phi}{\partial\omega}. \qquad [138r, s, t]$$

The Laplace equation, and the relations between ϕ and ψ if ψ exists, are

$$\frac{\partial}{\partial\zeta}\left[(\zeta^2 + 1)\frac{\partial\phi}{\partial\zeta}\right] + \frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial\phi}{\partial\mu}\right] + \frac{\zeta^2 + \mu^2}{(\zeta^2 + 1)(1-\mu^2)}\frac{\partial^2\phi}{\partial\omega^2} = 0, \qquad [138u]$$

$$\frac{\partial\psi}{\partial\zeta} - k(1-\mu^2)\frac{\partial\phi}{\partial\mu}, \quad \frac{\partial\psi}{\partial\mu} = -k(\zeta^2+1)\frac{\partial\phi}{\partial\zeta}. \qquad [138v, w]$$

Suppose now, that a solid ellipsoid is given whose surface is defined by

$$\frac{x}{a^2} + \frac{y^2+z^2}{c^2} = 1, \quad c > a. \qquad [138x]$$

Then its ellipticity is $e = (c^2 - a^2)^{1/2}/c$; and for this ellipsoid $a' = a$, $c' = c$, so that, if on it $\zeta = \zeta_0$, from Equation [138j, k],

$$k = ec = \sqrt{c^2-a^2}, \quad \zeta_0 = a/k = a/\sqrt{c^2-a^2} = \sqrt{1-e^2}/e, \quad e = (\zeta_0^2+1)^{-1/2}, \qquad [138y, z, a']$$

and on this ellipsoid

$$x = a\mu, \quad \tilde{\omega} = c\sqrt{1-\mu^2}. \qquad [138b', c']$$

Five cases of the flow around such an ellipsoid or oblate spheroid will be treated. The general case can be handled by superposing flows of two or more of these five types.

Case 1. Oblate spheroid or Circular Disk, Moving Parallel to its Axis of Symmetry. Let its velocity be U toward $\mu = 1$; see Figure 233. Then

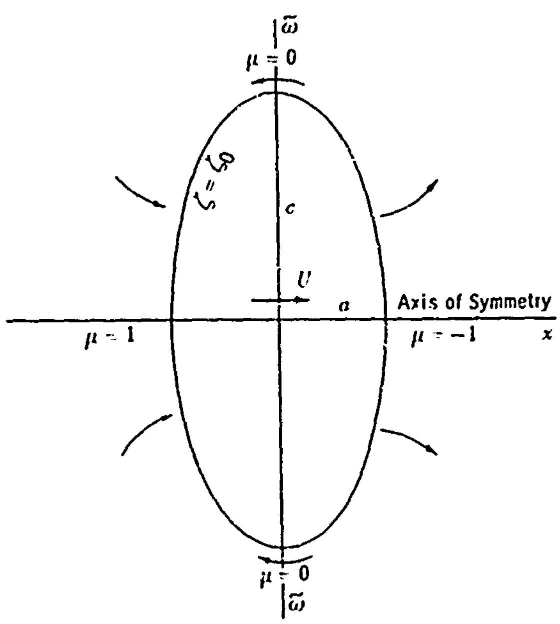


Figure 233 – Diagram for translation of an oblate spheroid in the direction of its axis.

$$\phi = g_2 k\, U \mu (1 - \zeta \cot^{-1} \zeta), \qquad [138d']$$

$$\psi = -\frac{1}{2} g_2 k^2\, U (\zeta^2 + 1)(1 - \mu^2) \left(\cot^{-1} \zeta - \frac{\zeta}{\zeta^2 + 1}\right), \qquad [138e']$$

$$g_2 = \left(\cot^{-1} \zeta_0 - \frac{\zeta_0}{\zeta_0^2 + 1}\right)^{-1} = \left(\sin^{-1} e - e\sqrt{1 - e^2}\right)^{-1}. \qquad [138f']$$

If $e = 1$, so that $\zeta_0 = 0$, $g_2 = 2/\pi$. As $e \to 0$, and $\zeta_0 \to \infty$, $e^3 g_2 \to 3/2$, as is easily verified by using Equation [138x'] and the series, obtained from Equation [33k] and valid for any real number $\xi \geq 1$,

$$\cot^{-1} \xi = \tan^{-1} \frac{1}{\xi} = \frac{1}{\xi} - \frac{1}{3\,\xi^3} + \frac{1}{5\,\xi^5} \ldots. \qquad [138g']$$

Toward infinity, with use of the last series,

$$\phi = \frac{1}{3} g_2 k\, U \frac{\mu}{\zeta^2} = \frac{1}{3} e^3 g_2 c^3\, U \frac{\cos \theta}{r^2}$$

approximately, since $k = ec$. Thus the flow is that of a dipole. As $e \to 0$ and $e^3 g_2 \to 3/2$, $\phi \to c^3\, U \cos \theta / 2r^2$, as for a sphere of radius c; see Section 127.

The velocity components in the coordinate directions are $q_\omega = 0$ and

$$q_\zeta = g_2\, U \mu \left(\frac{\zeta^2 + 1}{\zeta^2 + \mu^2}\right)^{1/2} \left(\cot^{-1} \zeta - \frac{\zeta}{\zeta^2 + 1}\right), \qquad [138h']$$

$$q_\mu = -g_2\, U \left(\frac{1 - \mu^2}{\zeta^2 + \mu^2}\right)^{1/2} \left(1 - \zeta \cot^{-1} \zeta\right). \qquad [138i']$$

On the x-axis, $\mu = \pm 1$, $x = \pm k\zeta$, $q = |u|$ and

$$u = \pm q_\zeta = g_2\, U \left(\cot^{-1} \frac{|x|}{k} - \frac{k|x|}{x^2 + k^2}\right). \qquad [138j']$$

On the equatorial or yz-plane, $\mu = 0$, $\varpi = k\sqrt{\zeta^2+1}$, $q = |u|$ and

$$u = q_\mu = -g_2 U\left(\frac{k}{\sqrt{\varpi^2 - k^2}} - \sin^{-1}\frac{k}{\varpi}\right); \tag{138k'}$$

in particular, on the circumference of the ellipsoid itself,

$$\zeta = \zeta_0,\ \varpi = k\sqrt{\zeta_0^2+1} = k/e \quad \text{and}$$

$$u = -g_2 U\left(\frac{e}{\sqrt{1-e^2}} - \sin^{-1} e\right) \tag{138l'}$$

For a figure, see Figure 234 as explained a little later.

The kinetic energy of the fluid, found by the method that was employed in obtaining Equation [137f'], is

$$T = \frac{2}{3}\pi\rho c^3 g_2 U^2\left(e - \sqrt{1-e^2}\ \sin^{-1} e\right) \tag{138m'}$$

Circular Disk

If $\zeta_0 = 0$, $e = 1$, $a = 0$ and the ellipsoid becomes a circular disk of radius $c = k$ moving perpendicularly to its surface. Then $g_2 = 2/\pi$. On the disk $\varpi = c(1-\mu^2)^{1/2}$ and $\psi = -U\varpi^2/2$. Also, on its front face, $q_\zeta = u = U$ and

$$q_\varpi = -q_\mu = \frac{2U}{\pi\mu}\sqrt{1-\mu^2} = \frac{2U}{\pi}\,\frac{\varpi}{(c^2-\varpi^2)^{1/2}}. \tag{138n'}$$

Here μ increases inward, ϖ outward, hence the negative sign. At the edge $q_\varpi \to \infty$. On the rear face $\mu = -(1-\varpi^2/c^2)^{1/2}$ and the velocity is reversed.

The kinetic energy of the fluid is, from Equation [138m'],

$$T = \frac{4}{3}\rho c^3 U^2. \tag{138o'}$$

Some lines of flow near a moving circular disk, drawn for equidistant values of ψ, are shown in Figure 234.

The same diagram will serve to illustrate the motion outside of any oblate spheroid placed so that its focal circle coincides with the perimeter of the disk, such as the one shown in outline by the elliptical curve in Figure 234. The ellipsoid is assumed to be in translation along its axis of symmetry. For, if k is fixed, variation of ζ_0 changes only the factor of proportionality g_2 in ψ, which changes ψ at all points in the same ratio but does not alter the geometrical pattern of the streamlines.

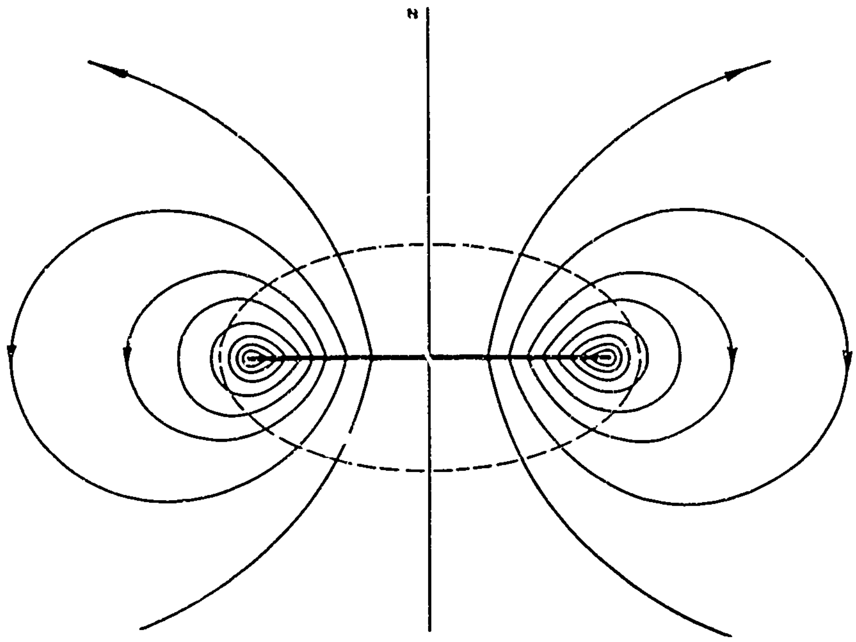


Figure 234 – See Section 138, Case 1. (Copied from Reference 1.)

Case 2. Flow Past an Oblate Spheroid or a Circular Disk, Parallel to its Axis. Let the velocity of the fluid at infinity be U toward $\mu = -1$. Adding to the expressions for ϕ and ψ in Equations [138d′, e′] Ux for ϕ and $U\varpi^2/2$ for ψ,

$$\phi = kU\mu[\zeta + g_2(1 - \zeta \cot^{-1} \zeta)], \tag{138p'}$$

$$\psi = \frac{1}{2} k^2 U (\zeta^2 + 1)(1 - \mu^2) \left[1 - g_2 \left(\cot^{-1} \zeta - \frac{\zeta}{\zeta^2 + 1}\right)\right] \tag{138q'}$$

Here again $q_\varpi = 0$; and, if a prime denotes values as given by Equations [138h′ to 138k′],

$$q_\zeta = q_\zeta' - U\mu \left(\frac{\zeta^2 + 1}{\zeta^2 + \mu^2}\right)^{1/2}, \quad q_\mu = q_\mu' - U\zeta \left(\frac{1 - \mu^2}{\zeta^2 + \mu^2}\right)^{1/2} \tag{138r', s'}$$

On the x-axis and in the yz-plane $q = |u|$ and $u = u' - U$.

On the ellipsoid itself, where $\zeta = \zeta_0$, using the value of g_2, $\psi = 0$, $q_\zeta = 0$, hence $q = |q_\mu|$, and, after combining terms,

$$q_\mu = -\frac{g_2 U}{\zeta_0^2+1}\left(\frac{1-\mu^2}{\zeta_0^2+\mu^2}\right)^{1/2} = -e^3 g_2 U \frac{\tilde{\omega}}{\sqrt{c^2-e^2\tilde{\omega}^2}} \qquad [138t']$$

from Equations [138y, z] and Equation [138d].

A circular disk is obtained again by setting $e = 1$, $\zeta_0 = 0$. Then on the disk $q = |q_{\tilde{\omega}}|$ where $q_{\tilde{\omega}}$ equals $\mp q_\mu$ and is again given by Equation [138n']. In steady motion the excess of pressure at points on the disk above that at infinity is

$$p - p_\infty = \frac{1}{2}\rho(U^2-q^2) = \frac{1}{2}\rho U^2\left(1-\frac{4}{\pi^2}\frac{\tilde{\omega}^2}{c^2-\tilde{\omega}^2}\right), \qquad [138u']$$

Thus $p = p_\infty$ at $\tilde{\omega} = 0.844\,c$.

Streamlines selected to be equidistant at infinity are shown in Figure 235 for an ellipsoid with $\zeta_0 = 0.577$, $e = 0.866$, $g_2 = 1.628$, and for a disk in Figure 236. For the ellipsoid, $p - p_\infty$ is shown on an arbitrary scale; it vanishes at $\mu = 0.68$, $\tilde{\omega} = 0.73\,c$. Values of $p - p_\infty$ are shown along the x-axis and along the ellipsoid, also, plotted horizontally, along the $\tilde{\omega}$-axis above the ellipsoid.

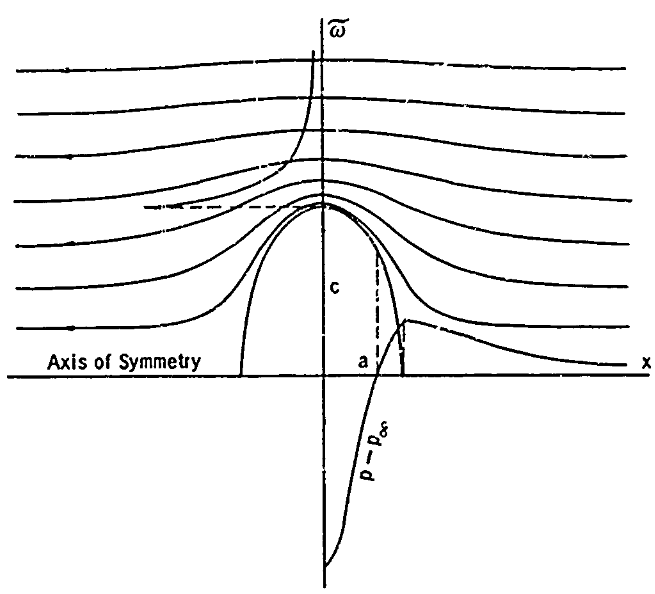


Figure 235 – Streamlines for flow past an oblate spheroid in the direction of its axis of symmetry. The distribution of pressure p is shown along the axis, then over the spheroid, and outward along a transverse axis. See Section 138, Case 2.

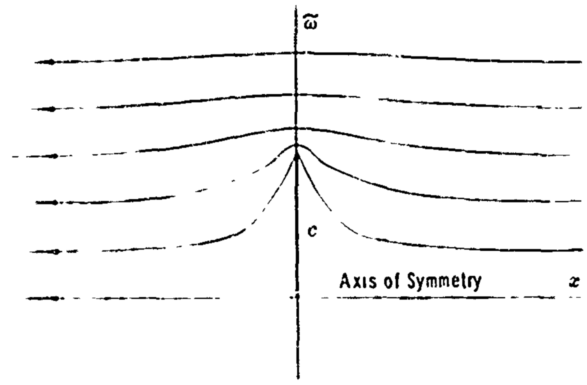


Figure 236 – Flow past a circular disk. See Section 138, Case 2.

Case 3. Motion of an Oblate Spheroid Perpendicular to its Axis of Symmetry. At velocity V toward positive y, as in Figure 237, if $\zeta = \zeta_0$ on the ellipsoid,

$$\phi = h_2 k V (\zeta^2 + 1)^{1/2} (1 - \mu^2)^{1/2} \left(\cot^{-1} \zeta - \frac{\zeta}{\zeta^2 + 1} \right) \cos \omega, \qquad [138v']$$

$$h_2 = \left(\frac{\zeta_0^2 + 2}{\zeta_0(\zeta_0^2 + 1)} - \cot^{-1} \zeta_0 \right)^{-1} = \left(\frac{e + e^3}{\sqrt{1 - e^2}} - \sin^{-1} e \right)^{-1}. \qquad [138w']$$

As $e \to 0$ and $\zeta_0 \to \infty$, $e^3 h_2 \to 3/4$, as appears from the series $(1 - e^2)^{-1/2} = 1 + e^2/2 + \ldots$ and the series for $\sin^{-1} e$ as obtained from Equation [33j]. As $e \to 1$ and $\zeta_0 \to 0$, $h_2 \to 0$. Thus ϕ equals 0 for a disk, as it must.

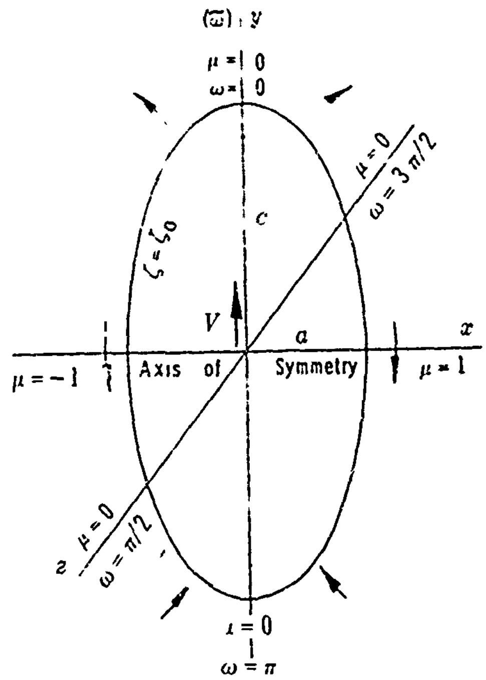
Toward infinity, $\zeta \to r/k = r/ec$ and, using Equation [138g′] and the series

$$\frac{\zeta}{\zeta^2 + 1} = \frac{1}{\zeta} \left(1 + \frac{1}{\zeta^2} \right)^{-1} = \frac{1}{\zeta} - \frac{1}{\zeta^3} \ldots , \qquad [138x']$$

$$\phi = \frac{2}{3} h_2 k V (1 - \mu^2)^{1/2} \frac{\cos \omega}{\zeta^2} = \frac{2}{3} e^3 h_2 c^3 V \frac{y}{r^3} \qquad [138y']$$

approximately, which is the potential of a dipole with its axis parallel to y. As $\zeta_0 \to \infty$, and $e \to 0$, since $e^3 h_2 \to 3/4$, $\phi \to c^3 \, Vy/2r^3$, as for a moving sphere of radius c.

Figure 237 -- Diagram for translation of an oblate spheroid parallel to an equatorial axis. Dots indicate traces of a focal circle. See Section 138, Case 3.



The components of velocity in the coordinate directions are, from Equations [138r, s, t],

$$q_\zeta = h_2 V \left(\frac{1-\mu^2}{\zeta^2+\mu^2}\right)^{1/2} \left(\frac{\zeta^2+2}{\zeta^2+1} - \zeta \cot^{-1} \zeta\right) \cos \omega, \qquad [138z']$$

$$q_\mu = h_2 V \mu \left(\frac{\zeta^2+1}{\zeta^2+\mu^2}\right)^{1/2} \left(\cot^{-1} \zeta - \frac{\zeta}{\zeta^2+1}\right) \cos \omega, \qquad [138a'']$$

$$q_\omega = h_2 V \left(\cot^{-1} \zeta - \frac{\zeta}{\zeta^2+1}\right) \sin \omega. \qquad [138b'']$$

On the y-axis and on the zx-plane the velocity is in the direction of $\pm V$ and $q = |v|$. On the y-axis $\cos \omega = \pm 1$, $\mu = 0$, $y = \pm k(\zeta^2+1)^{1/2}$, $\zeta = (y^2/k^2 - 1)^{1/2}$, and

$$v = \pm q_\zeta = h_2 V \left(\frac{k(y^2+k^2)}{y^2\sqrt{y^2-k^2}} - \sin^{-1} \frac{k}{|y|}\right). \qquad [138c'']$$

On the x-axis $x = \pm k\zeta$ and, from the limit of $-q_\mu$ as $\mu \to \pm 1$ with $\omega = 0$,

$$v = -h_2 V\left(\cot^{-1}\frac{|x|}{k} - \frac{k|x|}{x^2 + k^2}\right). \qquad [138d'']$$

On the z-axis $\mu = 0$, $\sin\omega = \pm 1$, $z = \pm k(\zeta^2 + 1)^{1/2}$, and

$$v = \mp q_\omega = -h_2 V\left(\sin^{-1}\frac{k}{|z|} - \frac{k\sqrt{z^2 - k^2}}{z^2}\right) \qquad [138e'']$$

Over the surface of the ellipsoid itself, on which $\zeta = \zeta_0$ and is constant, the relative variation of q_ω is similar to that over a moving sphere. Furthermore, around the circumference in the transverse or zx-plane, the velocity q is uniform, since $|\sin\omega| = 1$, and, using Equations [138z, a'],

$$q = |q_\omega| = h_2 V\left(\sin^{-1} e - e\sqrt{1 - e^2}\right). \qquad [138f'']$$

The kinetic energy of the fluid, found by the method employed in obtaining Equation [137f'], is

$$T = \frac{2}{3}\pi\rho h_2 c^3 V^2 \sqrt{1 - e^2}\left(\sin^{-1} e - e\sqrt{1 - e^2}\right). \qquad [138g'']$$

Case 4. Flow Past an Oblate Spheroid Perpendicular to its Axis of Symmetry. Let the fluid at infinity flow toward negative y at velocity V. Then, adding Vy in ϕ,

$$\phi = kV(\zeta^2 + 1)^{1/2}(1 - \mu^2)^{1/2}\left[1 + h_2\left(\cot^{-1}\zeta - \frac{\zeta}{\zeta^2 + 1}\right)\right]\cos\omega. \qquad [138h'']$$

If a prime denotes values given by Equations [138z', a'', b''],

$$q_\zeta = q_\zeta' - V\zeta\left(\frac{1 - \mu^2}{\zeta^2 + \mu^2}\right)^{1/2}\cos\omega, \quad q_\mu = q_\mu' + V\mu\left(\frac{\zeta^2 + 1}{\zeta^2 + \mu^2}\right)^{1/2}\cos\omega, \qquad [138i'', j'']$$

$$q_\omega = q'_\omega + V \sin \omega. \quad [138k'']$$

On the y-axis and in the zx-plane $q = |v|$ and $v = v' - V$.

On the ellipsoid itself, where $\zeta = \zeta_0$, $q_\zeta = 0$ and, using Equation [138w′] and Equations [138z, a′],

$$q_\mu = \frac{2e^3 h_2 V}{(1-e^2)^{1/2}} \frac{\mu \cos \omega}{(1-e^2+e^2\mu^2)^{1/2}} \quad [138l'']$$

$$q_\omega = \frac{2e^3 h_2 V}{(1-e^2)^{1/2}} \sin \omega. \quad [138m'']$$

The same remark concerning the variation of q_ω applies here as in Case 3. Around the circumference of the ellipsoid in the zx-plane, $q = |q_\omega|$ and q_ω is given by Equation [138m″] with $\sin \omega = 1$. As $e \to 0$, the coefficient $2e^3 h_2/(1-e^2)^{1/2} \to 3/2$, as for a moving sphere.

Case 5. Rotation of an Oblate Spheroid or a Circular Disk about an Equatorial Axis. Let the angular velocity be Ω about the y-axis. Then

$$\phi = A(\zeta^2+1)^{1/2} \mu(1-\mu^2)^{1/2} \left(3 - \frac{1}{\zeta^2+1} - 3\zeta \cot^{-1} \zeta\right) \sin \omega, \quad [138n'']$$

$$A = k^2\Omega \left[3(2\zeta_0^2+1) \cot^{-1} \zeta_0 - 6\zeta_0 - \frac{\zeta_0}{\zeta_0^2+1}\right]^{-1}. \quad [138o'']$$

At $\zeta = \zeta_0$ this satisfies the boundary condition stated in Equation [137c″], which is easily seen to hold for planetary coordinates as well. The axes are assumed to share in the rotation.

At large distances from the ellipsoid where ζ is large, it is found, by expanding in powers of $1/\zeta$ as in previous cases, using Equation [138g′], that approximately,

$$\phi = \frac{2}{5} \frac{A\mu}{\zeta^3} (1-\mu^2)^{1/2} \sin \omega = \frac{2}{5} e^3 c^3 A \frac{zx}{r^5}.$$

The velocity components are

$$q_\zeta = \frac{A}{k}\ \frac{\mu(1-\mu^2)^{1/2}}{(\zeta^2+\mu^2)^{1/2}} \left[3(2\zeta^2+1)\cot^{-1}\zeta - 6\zeta - \frac{\zeta}{\zeta^2+1}\right]\sin\omega, \qquad [138\text{p}'']$$

$$q_\mu = \frac{A}{k}\ \frac{1-2\mu^2}{(\zeta^2+\mu^2)^{1/2}}\,(\zeta^2 \quad 1)^{1/2}\left(3\zeta\cot^{-1}\zeta - 3 + \frac{1}{\zeta^2+1}\right)\sin\omega, \qquad [138\text{q}'']$$

$$q_\omega = \frac{A}{k}\,\mu\left(3\zeta\cot^{-1}\zeta - 3 + \frac{1}{\zeta^2+1}\right)\cos\omega. \qquad [138\text{r}'']$$

For a *circular disk,* obtained by letting $a \to 0$, so that $\zeta_0 \to 0$ and $k \to c$, $A = 2c^2\,\Omega/3\pi$, and on the disk itself $y = c(1-\mu^2)^{1/2}\cos\omega$, $z = c(1-\mu^2)^{1/2}\sin\omega$, and $c\mu = \pm(c^2-y^2-z^2)^{1/2}$. Thus, on the side on which $\mu > 0$,

$$\phi = \frac{4}{3\pi}\,\Omega\, z\,(c^2-y^2-z^2)^{1/2}, \qquad [138\text{s}'']$$

and the y and z components of velocity tangential to the disk are

$$v = -\frac{\partial\phi}{\partial y} = \frac{4\Omega}{3\pi}\ \frac{yz}{(c^2-y^2-z^2)^{1/2}}, \qquad [138\text{t}'']$$

$$w = -\frac{\partial\phi}{\partial z} = \frac{4\Omega}{3\pi}\ \frac{y^2+2z^2-c^2}{(c^2-y^2-z^2)^{1/2}}. \qquad [138\text{u}'']$$

On the opposite side of the disk ϕ , v, and w are reversed in sign.

Over most of the disk the fluid flows rather as if to go round the axis in the direction of rotation of the disk. Close to the edge the values of u and v are such that the radial component of the velocity predominates, becoming infinite at the edge. and its direction is that of a flow around the edge in opposition to the rotation.

The kinetic energy of the surrounding fluid of density ρ is

$$T = \frac{8}{45}\,\rho c^5\Omega^2. \qquad [138\text{w}'']$$

For, the velocity normal to the disk is that of the disk itself or $q_n = \Omega z$; hence writing $y = \varpi\cos\omega$, $z = \varpi\sin\omega$, and $dS = \varpi d\varpi d\omega$, Equation [17c] gives, integrated over both faces,

$$T = \frac{1}{2}\rho\int\phi q_n dS = \rho\int_0^c \frac{4}{3\pi}\,\Omega^2\varpi^3(c^2-\varpi^2)^{1/2}d\varpi\int_0^{2\pi}\sin^2\omega\,d\omega.$$

The pressure can be found from Equation [11c] or Equation [11d].

In the last three cases there is no axis of symmetry, hence no stream function exists. (See Reference 1, Article 107, 109; Reference 2, Section 15.54, 15.55; Zahm.[102,174])

139. CIRCULAR APERTURE

The oblate-spheroidal coordinates described in the last section may be used also to obtain the flow through a circular aperture in an infinite plane.

For this purpose k is taken equal to c, the radius of the aperture, so that from Equations [138a, b]

$$x = c\mu\zeta,\quad \varpi = c(\zeta^2+1)^{1/2}\ (1-\mu^2)^{1/2}; \qquad [139a, b]$$

and the ranges, $-\infty<\zeta<\infty$, $0\leqq\mu\leqq 1$ are used, so that, as in Figure 238, $\mu = 1$ on the entire x-axis.

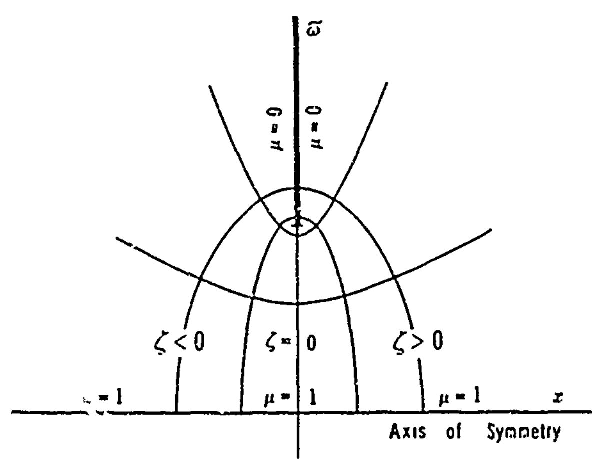


Figure 238 – Flow through a circular aperture in an infinite plane using oblate-spheroidal coordinates.

In this way the discontinuity in the coordinates on the plane $x = 0$ is displaced to the outlying part where $\varpi > k = c$, which is now a rigid boundary; the two halves of each coordinate ellipsoid meet there with opposite values of ζ. The central part of the plane, on which $\varpi < k = c$, represents a circular aperture. In the last section, discontinuities of μ were allowed to occur on the central disk of radius k, but this part was there enclosed in a rigid body. Thus in each case continuity of coordinates is preserved throughout the entire space occupied by fluid

It will be found that the differential Equations [138u] and [138v, w] are satisfied by

$$\phi = A \cot^{-1} \zeta, \quad \psi = A c \mu. \qquad [139c, d]$$

The velocity is in the ζ-direction, so that $q = |q_\zeta|$, and from Equations [138r, n]

$$q_\zeta = \frac{A}{c} (\zeta^2 + 1)^{-1/2} (\zeta^2 + \mu^2)^{-1/2} \qquad [139e]$$

On either face of the plane boundary, $\mu = 0$, $x = 0$ and $\varpi = c(\zeta^2 + 1)^{1/2}$ or $\zeta = \pm \sqrt{\varpi^2 - c^2}/c$; hence $q = |q_\varpi|$ and

$$q_\varpi = \pm q_\zeta = \frac{A}{c\zeta} (\zeta^2 + 1)^{-1/2} = \frac{\pm cA}{\varpi \sqrt{\varpi^2 - c^2}}. \qquad [139f]$$

The sign $\pm$ is to be taken the same as the sign of ζ.

In the plane of the opening $\zeta = 0$, $x = 0$, $\varpi = c(1 - \mu^2)^{1/2}$, $q = |u|$, and

$$u = q_\zeta = \frac{A}{c\mu} = \frac{A}{\sqrt{c^2 - \varpi^2}}. \qquad [139g]$$

On the axis of symmetry or x-axis, $\mu = 1$, $x = c\zeta$, $q = |u|$ and

$$u = q_\zeta = \frac{A}{c(\zeta^2 + 1)} = \frac{cA}{x^2 + c^2}. \qquad [139h]$$

The velocity is thus infinite at the edge of the opening.

The volume of fluid flowing per second through the opening is, using Equation [139g] for u,

$$Q = 2\pi \int_0^c u \varpi d\varpi = 2\pi \; c.1. \qquad [139i]$$

Some lines of flow for equidistant values of ψ are shown in Figure 239. By adding a uniform flow parallel to the plane, more general cases can be treated.

(See Reference 1, Article 108.)

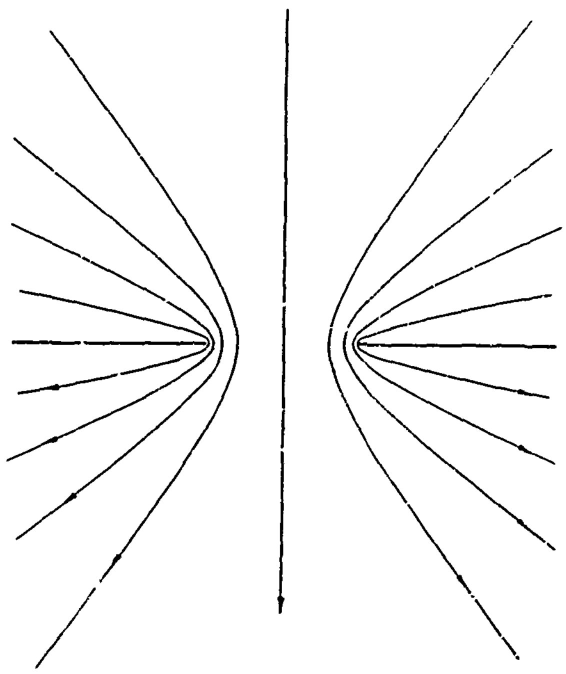


Figure 239 – Symmetrical streamlines for flow through a circular aperture in an infinite plane wall.

140. ROTATING ELLIPSOIDAL SHELL

Consider the fluid inside a shell whose surface is the ellipsoid

$$f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0. \qquad [140a]$$

The direction cosines of the normal at any point of the surface are, from Equations [136b, c, d],

$$l = 2k \frac{x}{a^2}, \quad m = 2k \frac{y}{b^2}, \quad n = 2k \frac{z}{c^2}.$$

Let the ellipsoid rotate about the x-axis with angular velocity ω_x. Then, by substituting for l, m, n in Equation [135d], and also $u = -\partial\phi/\partial x$, $v = -\partial\phi/\partial y$, $w = -\partial\phi/\partial z$, $\omega_y = \omega_z = 0$, the boundary condition to be satisfied by ϕ is

$$-\frac{x}{a^2}\frac{\partial\phi}{\partial x} - \frac{y}{b^2}\frac{\partial\phi}{\partial y} - \frac{z}{c^2}\frac{\partial\phi}{\partial z} = \omega_x\left(\frac{1}{c^2} - \frac{1}{b^2}\right) yz.$$

A solution of the last equation which is also a solution of the Laplace Equation or Equation [7a] is

$$\phi = -\frac{b^2 - c^2}{b^2 + c^2}\,\omega_x\, yz. \qquad [140b]$$

The components of velocity are

$$u = 0, \quad v = \frac{b^2 - c^2}{b^2 + c^2}\,\omega_x\, z, \quad w = \frac{b^2 - c^2}{b^2 + c^2}\,\omega_x\, y. \qquad [140c, d, e]$$

The flow thus proceeds in planes perpendicular to the x-axis, and it is the same in all of these planes, except for variation in the size of the occupied elliptical cross section. The flow pattern is, in fact, the same as that inside an elliptical cylinder rotating about its axis, as illustrated in Figure 173.

The kinetic energy of the fluid is

$$T = \frac{\rho\omega_x^2}{2}\left(\frac{b^2 - c^2}{b^2 + c^2}\right)^2 \iiint (y^2 + z^2)\, dx\, dy\, dz = \frac{2\pi}{15}\,\rho\, abc\,\frac{(b^2 - c^2)^2}{b^2 + c^2}\,\omega_x^2. \qquad [140f]$$

To evaluate the integral, substitute $x = ax'$, $y = br'\cos\theta$, $z = cr'\sin\theta$.

Analogous results hold for rotation about the y- or z-axes; and by combining rotations about the coordinate axes the general case can be represented of rotation about any axis passing through the center of the ellipsoid.

The axes rotate, of course, with the shell.

(See Reference 1, Article 110; Zahm.[174])

141. ELLIPSOID WITH UNEQUAL AXES

For the general ellipsoid with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \qquad [141a]$$

the appropriate ellipsoidal coordinates λ_1, λ_2, λ_3 are defined in terms of x, y, z as the three roots of a cubic in λ, which can be written

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} = 1. \qquad [141b]$$

The use of these orthogonal coordinates in potential problems involves special functions known as Lame´ functions, and no further details will be given here.

For translation of the ellipsoid through fluid at rest at infinity, parallel to one of its axes, which will be taken as the a-axis without regard to its relative magnitude, the kinetic energy of the fluid is found to be

$$T = \frac{\alpha_0}{2-\alpha_0} \cdot \frac{2}{3} \pi\, abc\, \rho\, U^2, \qquad [141c]$$

$$\alpha_0 = abc \int_0^\infty \frac{d\lambda}{(a^2+\lambda)^{3/2}\,(b^2+\lambda)^{1/2}\,(c^2+\lambda)^{1/2}}. \qquad [141d]$$

The definite integral can be expressed in terms of elliptic integrals, which are tabulated; see Reference 3 or Reference 235, as listed later, where $a > b > c$.

For rotation at angular velocity ω about an axis, here taken as the a-axis,

$$T = \frac{(b^2-c^2)^2\,(\gamma_0-\beta_0)}{2(b^2-c^2)+(b^2+c^2)\,(\beta_0-\gamma_0)} \cdot \frac{2}{15} \pi\, abc\, \rho\, \omega^2, \qquad [141e]$$

$$\beta_0 = abc \int_0^\infty \frac{d\lambda}{(a^2+\lambda)^{1/2}\,(b^2+\lambda)^{3/2}\,(c^2+\lambda)^{1/2}}, \qquad [141f]$$

$$\gamma_0 = abc \int_0^\infty \frac{d\lambda}{(a^2+\lambda)^{1/2}\,(b^2+\lambda)^{1/2}\,(c^2+\lambda)^{3/2}}. \qquad [141g]$$

If one axis is reduced to zero, the ellipsoid becomes an elliptical disk.

(See Reference 1, Articles 112-115; Durand,[3] Volume I, p. 293; Tuckerman;[235] Zahm.[174])

142. ELLIPSOID CHANGING SHAPE

If the semiaxes of the ellipsoid defined by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \qquad [142a]$$

change with time at the rates $\dot{a}$, $\dot{b}$, $\dot{c}$, without rotation of the ellipsoid and with its center at rest, and if a point moves with components of velocity $\dot{x}$, $\dot{y}$, $\dot{z}$, in such a way as to remain always on the surface of the ellipsoid, then at this point Equation [142a] is always satisfied, and, differentiating Equation [142a] with respect to the time,

$$\frac{x}{a^2}\dot{x} + \frac{y}{b^2}\dot{y} + \frac{z}{c^2}\dot{z} - \frac{x^2}{a^3}\dot{a} - \frac{y^2}{b^3}\dot{b} - \frac{z^2}{c^3}\dot{c} = 0. \qquad [142b]$$

Now according to Equations [135f, g, h] the direction cosines of the normal to the ellipsoid at the point x, y, z are

$$l = 2k\frac{x}{a^2}, \quad m = 2k\frac{y}{b^2}, \quad n = 2k\frac{z}{c^2},$$

where k is a constant of proportionality. After substituting in Equation [142b], the combination $l\dot{x} + m\dot{y} + n\dot{z}$ occurs; this represents the normal component of the velocity of the surface, which must equal the same component of the fluid velocity or

$$lu + mv + nw = -\left(l\frac{\partial\phi}{\partial x} + m\frac{\partial\phi}{\partial y} + n\frac{\partial\phi}{\partial z}\right)$$

in terms of the velocity potential ϕ. Hence, after eliminating l, m, n from the last expression, Equation [142b] gives for the boundary condition for ϕ

$$\frac{x}{a^2}\frac{\partial\phi}{\partial x} + \frac{y}{b^2}\frac{\partial\phi}{\partial y} + \frac{z}{c^2}\frac{\partial\phi}{\partial z} + \frac{x^2}{a^3}\dot{a} + \frac{y^2}{b^3}\dot{b} + \frac{z^2}{c^3}\dot{c} = 0. \qquad [142c]$$

This equation is satisfied by

$$\phi = -\frac{1}{2}\left(\frac{\dot{a}}{a}x^2 + \frac{\dot{b}}{b}y^2 + \frac{\dot{c}}{c}z^2\right), \qquad [142d]$$

which is a solution of the Laplace Equation or Equation [7a] provided

$$\frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c} = 0.$$

But this is merely the condition that the volume of the fluid enclosed in the ellipsoid or $4\pi\, abc/3$ shall remain constant, so that $(d/dt)\,[\log_e (abc)] = 0$.

The velocity components of the fluid are

$$u = -\frac{\partial\phi}{\partial x} = \frac{\dot{a}}{a}x, \quad v = \frac{\dot{b}}{b}y, \quad w = \frac{\dot{c}}{c}z. \qquad [142e, f, g]$$

(See Reference 1, Article 110.)

143. FLOW PAST A PARABOLOID

Consider the steady flow parallel to the axis of a solid body having the form of a paraboloidal solid of revolution. With the origin at its focus and the x-axis of cylindrical coordinates along its axis, let the equation of the surface of the solid be

$$\varpi^2 = a^2 - 2ax, \qquad [143a]$$

where ϖ denotes distance from the axis. Its apex is at $x = a/2$ and it extends toward negative x. Let the fluid approach at velocity U from $x = +\infty$.

It is convenient to introduce also parabolic coordinates λ_1, λ_2, nearly as in Section 87, so that

$$x = \frac{1}{2}(\lambda_2^2 - \lambda_1^2), \quad \varpi = \lambda_1\lambda_2, \quad \lambda_1 \geqq 0, \quad \lambda_2 \geqq 0;$$

$$\lambda_1 = \sqrt{r-x}, \quad \lambda_2 = \sqrt{r+x}, \quad r = \sqrt{x^2+\varpi^2} \qquad [143b, c, d]$$

The surfaces, λ_1 = constant or λ_2 = constant, are confocal paraboloids opening, respectively, toward positive and negative x. Their traces on a plane through the axis are illustrated in Figure 137. On the x-axis, $\lambda_1 = 0$ and $x = \lambda_2^2/2$ for $x \geqq 0$, whereas $\lambda_2 = 0$ and $x = -\lambda_1^2/2$ for $x \leqq 0$.

By introducing also for the moment $y = \varpi \cos \omega$ and $z = \varpi \sin \omega$ where ω is the angle about the axis, it is found from Equation [136a] that

$$\frac{\delta s_{\lambda_1}}{\delta\lambda_1} = \frac{\delta s_{\lambda_2}}{\delta\lambda_2} = (\lambda_1^2+\lambda_2^2)^{1/2}, \quad \frac{\delta s_\omega}{\delta\omega} = \frac{1}{\lambda_1\lambda_2} = \frac{1}{\varpi} \qquad [143e, f]$$

The surface of the given solid, on which $x<a$, is the paraboloid $\lambda_2 = \sqrt{a}$; as is easily verified from Equation [143c]. On this surface the stream function ψ must be constant. Furthermore, in the surrounding space, as $\lambda_2 \to \infty$, $x \to \infty$, and in the limit it is necessary that $\psi \quad U\varpi^2/2 = U\lambda_1^2\lambda_2^2/2$; see Equation [119b] for a uniform stream. The differential Equation [136m] for ψ becomes here

$$\frac{\partial}{\partial\lambda_1}\left(\frac{1}{\lambda_1\lambda_2}\frac{\partial\psi}{\partial\lambda_1}\right) + \frac{\partial}{\partial\lambda_2}\left(\frac{1}{\lambda_1\lambda_2}\frac{\partial\psi}{\partial\lambda_2}\right) = 0. \qquad [143g]$$

This equation and the two boundary conditions are satisfied if

$$\psi = \frac{1}{2}U\lambda_1^2(\lambda_2^2-a) = \frac{1}{2}U[\varpi^2 - a(r-x)]. \qquad [143h]$$

Then $\psi = 0$ on the solid and also where $x>0$ on the x-axis.

The corresponding potential, obtained by integrating Equations [136k, l], in which the lower signs are to be taken if $\lambda = \lambda_1$, $\mu = \lambda_2$, and the components and magnitude of the velocity are

$$\phi = \frac{1}{2} U(\lambda_2^2 - \lambda_1^2) - aU \ln \lambda_2 = Ux - \frac{1}{2} aU \ln(r + x), \qquad [143i]$$

$$q_x = U\left(-1 + \frac{a}{2r}\right), \quad q_{\varpi} = \frac{U}{2} \frac{a\varpi}{r(r+x)}, \quad q^2 = U^2\left(1 - \frac{a}{r} + \frac{a^2}{2r(r+x)}\right). \qquad [143j, k, l]$$

On the solid, since $r + x = \lambda_2^2 = a$, $q^2 = U^2(1 - a/2r)$.

The excess of pressure over that at infinity is everywhere positive. On the solid it is $p - p_\infty = \rho a U^2/4r$; on the x-axis ahead of it, where $r = x$,

$$p - p_\infty = \frac{1}{2} \rho U^2 \left(\frac{a}{x} - \frac{a^2}{4x^2}\right). \qquad [143m]$$

The streamlines are illustrated on half of a plane through the axis in Figure 240. The streamlines shown are equally spaced in the uniform stream and correspond to equal increments of ψ/ϖ. The value of $p - p_\infty$ at points along the axis and on the paraboloid is also shown, on an arbitrary scale and for steady motion.

(See Reference 2, Section 15.58.)

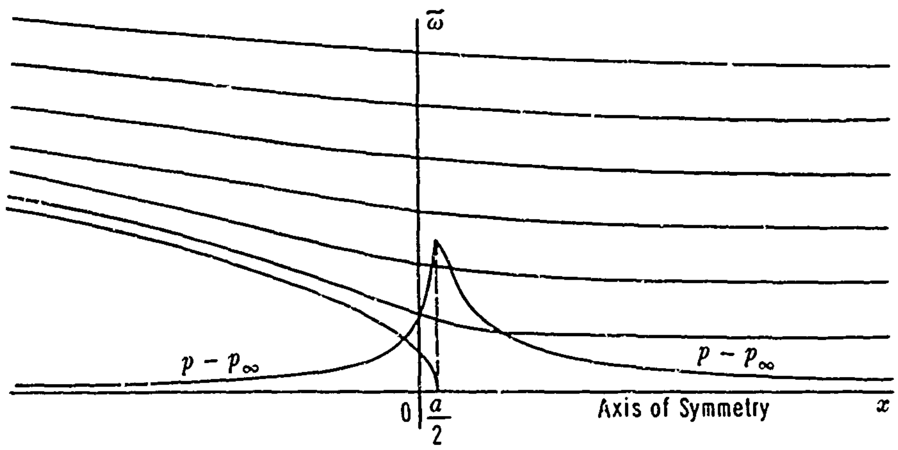


Figure 240 – Symmetrical flow past a paraboloid of revolution. The pressure along the axis and over the paraboloid is also shown. The focus is at 0. See Section 143.

144. AXISYMMETRIC JETS

A few cases of axisymmetric jets have been worked out by elaborate methods of approximation. A weakly contracting jet was treated by Reissner.[236]

Schack studied a round jet falling normally upon a plate;[237] his plot of the flow net, labeled in terms of the older convention as to the signs of the velocity potential ϕ and the stream function ψ, is reproduced in Figure 241. The figure shows half of a plane through the axis OD of the jet; the plate lies along OA produced both ways.

A jet issuing through a round hole in the infinite plane bottom of a tank, under the influence of internal pressure but not of gravity, was studied by Trefftz.[238] His plot of the flow net is reproduced in Figure 242; it also is labeled in terms of the older convention as to the signs of ϕ and ψ. The axis of the hole lies along the vertical line at A.

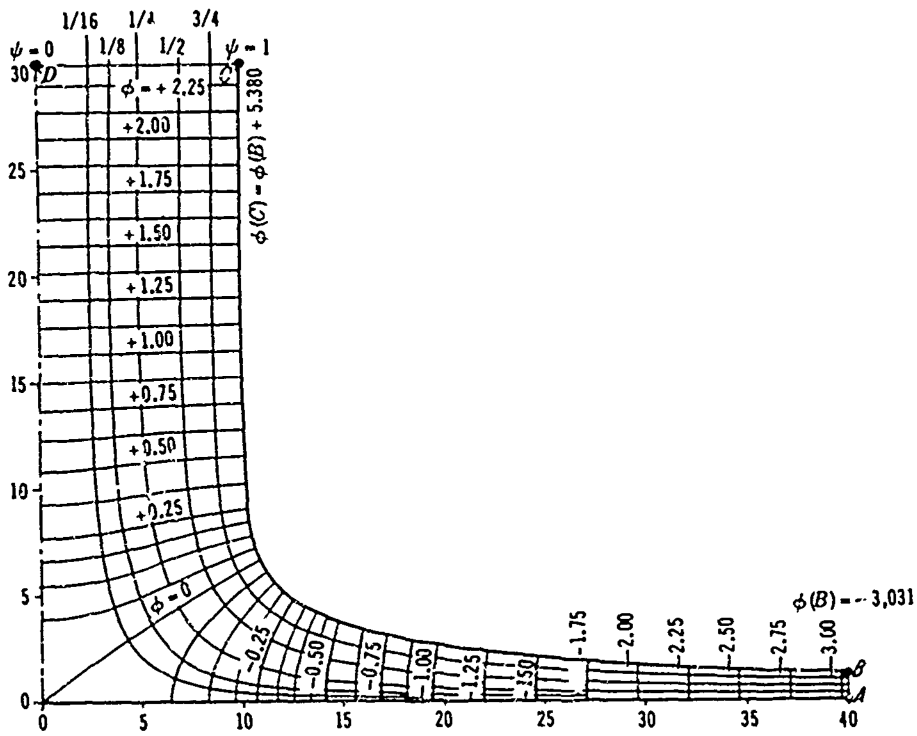


Figure 241 – Some streamlines and traces of equipotential surfaces, in a plane through the axis OD, for a round jet of fluid striking a rigid plate OA. The surface of the fluid is at CB. See Section 144. (Copied from Reference 237.)

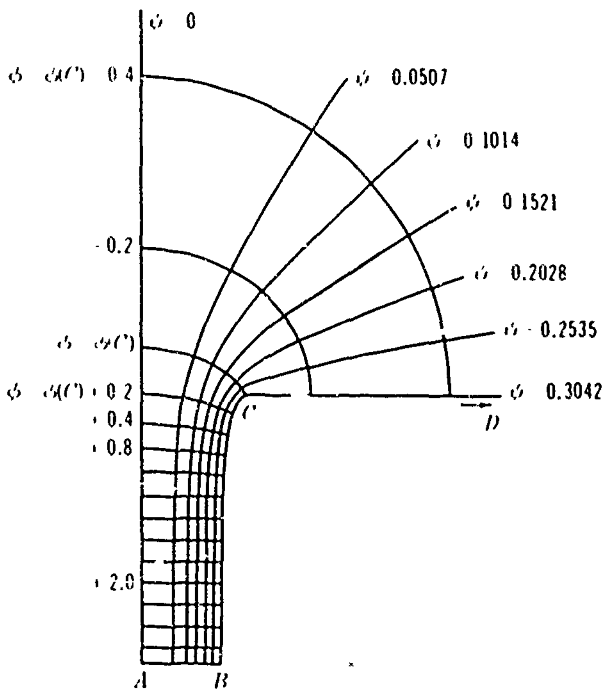


Figure 242 – Some streamlines and traces of equipotential surfaces in a plane through the axis at A, for fluid issuing through a round hole in an infinite plane wall, shown in part as CD. The surface of the issuing jet lies along CB. See Section 144. (Copied from Reference 238.)

145. OTHER THREE-DIMENSIONAL CASES

The following cases of relative motion between bodies and fluid have been treated:

(a) A spherical bowl, – see Basset,[5] I. p. 149;

(b) An anchor ring, – see references on p. 156 of Reference 2.

(c) Solid produced by revolving a limacon about its axis, by Bateman,[240] – see end of Chapter V.

(d) Solids of revolution in general, by Kaplan.[21]

(e) Certain special shapes of bodies by Koiossoff[241] and Greenhill.[242]

(f) Ellipsoid moving in a curved stream by Tallmien.[239]

(g) A point source on a sphere, by Masotti,[243] or on the axis of an oblate spheroid or a circular disk or near a round hole in a plate, by Nicholson;[244]

(h) A line vortex near a spheroid, by Poggi;[245]

(i) Motion of two spheroids, by Sen;[246]

(j) Two coaxial circular disks in a stream, by Sircar[247] and Nomura.[248] The disks repel each other in proportion to $\cos^2\phi$, where ϕ is the angle between the direction of the stream at infinity and the normal to the disks, and also experience torques tending to increase $\cos^2\phi$.

CHAPTER V

COEFFICIENTS OF INERTIA

146. EFFECTS OF FLUID INERTIA.

When a solid body submerged in an incompressible nonviscous fluid is acted upon by an external force, an acceleration is produced which is less than that for the same body in a vacuum, since the external force must also accelerate the fluid surrounding the body. It will be shown that the effect of the fluid can be represented by assigning to the solid an equivalent mass greater than the mass of the solid itself.

If the motion of the body is one of translation at velocity U in a fixed direction while the fluid is at rest at infinity, the velocity of the fluid at every point will be proportional to U and its kinetic energy will be proportional to U^2. Hence, the total kinetic energy of body and fluid can be expressed in the form

$$T = \frac{1}{2} (M + kM') U^2 \qquad [146a]$$

where M is the mass of the body,

M' is the mass of fluid displaced by it, and

k is a constant of proportionality called the *coefficient of inertia.*

The value of k will depend upon the size and shape of the body and, in general, upon its orientation relative to the direction of motion.

If F is the external force acting on the body in the direction of the motion, the rate at which F does work must be equal to the rate of increase of the total kinetic energy;

hence
$$FU = \frac{dT}{dt} = (M + kM') U \frac{dU}{dt}$$

and
$$F = (M + kM') \frac{dU}{dt} \qquad [146b]$$

Thus the acceleration produced is the same as if the mass of the body were increased from M to $M + kM'$. The added term kM' may be considered as an effective mass due to the presence of the fluid.

If U is constant, $F = 0$, so that no force is required to keep a body in motion provided there is no fluid friction. The coefficient of inertia has significance in cases of acceleration only.

For a body rotating about a fixed axis, similar considerations hold. The kinetic energy of body and fluid can be written

$$T = \frac{1}{2}(I + kI')\ \omega^2 \qquad [146c]$$

where I is the moment of inertia of the body about the axis of rotation,
I' is the moment of inertia of the displaced fluid when rotating as if solid, and
ω is the angular velocity.
The rate at which work is done then takes the form

$$G\omega = \frac{dT}{dt} = (I + kI')\ \omega\ \frac{d\omega}{dt},$$

where G is the torque acting on the body.

Hence
$$G = (I + kI')\ \frac{d\omega}{dt}. \qquad [146d]$$

This equation shows that no torque is required for constant angular rotation about a fixed axis in an ideal medium. The constant k is here the coefficient of inertia for rotation about the given axis, and its value for rotation is usually different from that for translation.

Two-dimensional flow, as described in Section 12 an important special case of fluid flow in which the motion occurs in a set of parallel planes, so that there is no component of velocity or acceleration in the direction perpendicular to these planes. In two-dimensional cases it will be understood that all quantities refer to the portion of the body and of the fluid that is contained between two planes drawn parallel to the planes of motion and unit distance apart, and T_1, M_1, M_1', I_1, I_1', will be written as referring to this portion. The coefficient of inertia, on the other hand, being merely a constant of proportionality, does not require a subscript.

It has been assumed that the motion is irrotational and is therefore entirely determined by the motion of the body. This assumption is essential. Furthermore, in defining the coefficient of inertia, only one component of the force or torque was considered; and the discussion was limited to certain special types of motion. It is of interest to consider how the coefficient of inertia may be used in certain other cases; and certain other features of the force action of fluids upon moving bodies may also be mentioned without proof.

For a given body of finite dimensions, with its mass distributed in any given manner, it can be shown that there is always at least one set of mutually perpendicular directions, fixed relative to the body, in any one of which the body can move through frictionless fluid without the action of any forces upon it and without exhibiting any tendency to rotate. These may be called directions of free translation. Directions perpendicular to a plane of geometrical symmetry for the surface of the body always have this property. As a general rule, if the body moves in any other direction, the fluid exerts a torque upon it, and this torque must be balanced by external forces if rotation is to be prevented. In cases of special symmetry there may be many directions of free translation; and some bodies, such as [illegible], can move freely in any direction.

A force applied in a direction of free translation produces acceleration in that direction only. A force applied in any other direction can be resolved into three perpendicular components, each of which acts in a direction of free translation. Each component will then produce a component of acceleration in its own direction, of the same magnitude as if the other components of force were absent; and the total acceleration will be the vector sum of the three component accelerations so produced. It the coefficients of inertia are different in the three directions, the resultant vector force and the resultant vector acceleration will not be parallel. As a simple example, to accelerate a massless thin disk through the fluid in a direction oblique to its plane, the applied force must necessarily be perpendicular to the plane of the disk.

To prevent rotational acceleration, it may be necessary also to apply a suitable torque.

Besides pure translation, other types of steady motion not requiring the application of external forces are possible. The most important case is that in which the surface of the body has two planes of symmetry and the line of intersection of these planes passes through the center of gravity of the body itself and is a principal axis of inertia for the body. Then a steady rotation is possible about that axis; and a torque applied about such an axis generates rotation about it in accord with the formula previously described. In special cases several or many such axes of free rotation may exist.

Two-dimensional motion may be further complicated by the presence of circulation about the body, which then necessarily has the form of an infinite cylinder. In translational motion the circulation gives rise to the familiar transverse force or lift; and the presence of circulation may make steady rotation of the cylinder impossible in the absence of external forces. Otherwise the statements that have been made for the three-dimensional case held also for two-dimensional motion.

In any case, the forces required to produce a given acceleration, translational or rotational, are independent of the motion already existing and are the same as if the fluid were at rest. This is easily seen from the pressure equation, as stated in Equation [9e]. Acceleration of the fluid motion is equivalent, at any time t_1, to the superposition upon the flow already existing at that time of an incremental flow that starts from rest. Since this added flow does not alter the velocities as they exist at time t_1, its only effect on the pressure at time t_1 is to add to the value of $\partial\phi/\partial t$ a term that depends upon the acceleration but not on the existing motion. It may happen that part of the total acceleration is actually due to hydrodynamical forces brought into play by the motion of the body through the fluid, such as the forces that have just been described; then the additional acceleration produced by the external forces is the same as it would be if these hydrodynamical accelerations were absent.

In the same way it can be seen that the forces required to produce two or more types of acceleration simultaneously are simply the vector sums of the forces required to produce each type separately.

In the last two chapters many expressions have been obtained for the kinetic energy, $kM'U^2/2$ or $kI'\omega^2/2$, of the fluid surrounding a moving or rotating body. To obtain k from these expressions, it is only necessary to introduce the known value of M' or I' and to divide the kinetic energy by $M'U^2/2$ or $I'\omega^2/2$. The values of the kinetic energy and of k are collected for convenience of reference in a table following the next section. A self-explanatory pictorial representation of the body is appended in each case.

147. NOTE ON UNITS.

In the formulas a consistent set of dynamical units is understood to be employed, as was explained in detail in Section 18. The coefficient of inertia k is a pure numeric.

To illustrate the use of the units the following problem will be solved.

An ellipsoidal body, with semiaxes 6 feet, 3 feet, and 3 feet, weighing 25,000 pounds, is suspended in sea water with its major axis vertical. When released, what will be its initial acceleration? The density of sea water is 64 pounds per cubic foot.

Solution: The resultant force on the body acts vertically downward and is equal to the weight of the body minus the buoyant force.

$$\text{Force} = 25{,}000 - \frac{4}{3}\pi \times 6 \times 3 \times 3 \times 64 = 10{,}500 \text{ pounds.}$$

$$\text{The mass of the body is } \frac{25{,}000}{32.2} = 776 \text{ slugs} = 776\ \frac{\text{lb-sec}^2}{\text{ft}}.$$

The coefficient of inertia for prolate spheroids moving "end on," with $a/b = 2.0$, is $k = 0.209$. Hence the effective mass of the fluid is

$$\frac{4}{3}\pi \times 6 \times 3 \times 3 \times \frac{64}{32.2} \times 0.209 = 94.0 \text{ slugs,}$$

and the total effective mass of body and fluid is 776 + 94.0 = 870 slugs. From Newton's second law of motion the acceleration is the resultant force divided by the mass or

$$\frac{10{,}500}{870} = 12.0 \text{ ft/sec}^2.$$

148. TABLE OF ENERGIES AND INERTIA COEFFICIENTS

a, b, c	Radius of a circle or semiaxis of an ellipse or ellipsoid, or half-width or width of a lamina
e	Ellipticity
k	Coefficient of inertia, a dimensionless constant

In translation, $k = \dfrac{\text{apparent increase in mass}}{\text{mass of displaced fluid}}$;

$$k = \frac{2T}{M'U^2} \quad \text{or} \quad \frac{2T_1}{M_1'U^2}.$$

In rotation, $k = \dfrac{\text{apparent increase in moment of inertia}}{\text{moment of inertia of displaced fluid}}$,

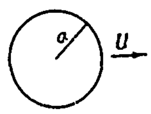
$$k = \frac{2T}{I'\omega^2} \quad \text{or} \quad \frac{2T_1}{I_1'\omega^2}.$$

I'	Moment of inertia of displaced fluid rotating as a rigid body about the assumed axis of rotation
I_1'	See under T_1
M'	Mass of fluid displaced by body
M_1'	See under T_1
T	Kinetic energy of fluid
T_1, I_1', M_1':	Values of T, I', M' for fluid between two planes parallel to the motion and unit distance apart, in cases of two-dimentional motion
U	Velocity of translation of body
θ	An angle in radians
ρ	Density of the fluid, in dynamical units
ω	Angular velocity of rotation of a body, in radians per second.

The fluid is assumed to surround the body and to be of infinite extent and at rest at infinity, except where other conditions are indicated. In regard to units, see Sections 18, 147.

A. TWO-DIMENSIONAL CASES

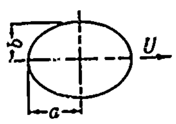
1. Circular cylinder in translation perpendicular to its axis:



$$T_1 = \frac{1}{2}\rho\pi\, a^2 U^2, \qquad \text{as in Equation [68i]},$$

$$M_1' = \rho\pi\, a^2, \qquad k = 1.$$

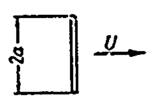
2. Elliptic cylinder in translation parallel to an axis, called the a-axis, either $a > b$ as shown or $b > a$:



$$T_1 = \frac{1}{2}\rho\pi\, b^2 U^2, \qquad \text{from Equation [84l]},$$

$$M_1' = \rho\pi\, ab, \qquad k = b/a.$$

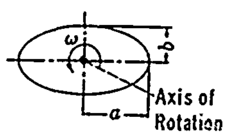
3. Plane lamina in translation perpendicular to its faces:



$$T_1 = \frac{1}{2}\rho\pi\, a^2 U^2, \qquad \text{as in Equation [86b]},$$

$$kM_1' = \rho\pi\, a^2.$$

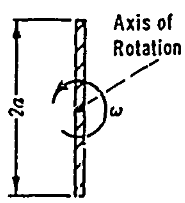
4. Elliptic cylinder rotating about its axis:



$$T_1 = \frac{1}{16}\rho\pi (a^2 - b^2)^2\, \omega^2, \qquad \text{as in Equation [106z]},$$

$$I_1 = \frac{1}{4}\rho\pi\, ab(a^2 + b^2), \qquad k = \frac{(a^2 - b^2)^2}{2ab\,(a^2 + b^2)}.$$

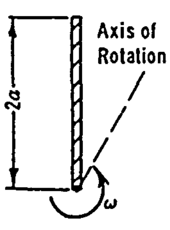
5. Plane lamina rotating about its central axis:



$$T_1 = \frac{1}{16}\rho\pi\, a^4 \omega^2, \qquad \text{as in Equation [106a']},$$

$$kI_1' = \frac{1}{8}\rho\pi\, a^4.$$

6. Plane lamina rotating about one edge:

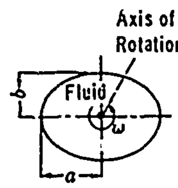


$$T_1 = \frac{9}{16}\rho\pi\, a^4 \omega^2, \qquad \text{as in Equation [106b']}$$

with $\beta = 1$,

$$\frac{\text{Apparent increase in moment of inertia}}{\text{Moment of inertia of fluid displaced by a cylinder of radius } a \text{ rotating as if rigid about a generator}} = \frac{9\rho\pi\, a^4/8}{3\rho\pi\, a^4/2} = \frac{3}{4}$$

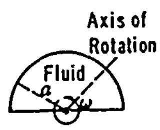
7. Fluid inside elliptic-cylindrical shell rotating about its axis:



$$T_1 = \frac{1}{8}\rho\pi ab\,\frac{(a^2-b^2)^2}{a^2+b^2}\,\omega^2, \quad \text{as in Equation [105m],}$$

$$I_1' = \frac{1}{8}\rho\pi\, ab\,(a^2+b^2), \qquad k = \left(\frac{a^2-b^2}{a^2+b^2}\right)^2.$$

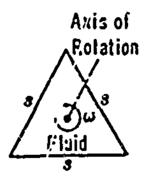
8. Fluid inside semicircular cylindrical shell rotating about axis of the semicircle



$$T_1 = \frac{\pi}{4}\left(\frac{8}{\pi^2}-\frac{1}{2}\right)\rho a^4\omega^2, \quad \text{as in Equation [102e],}$$

$$I_1' = \frac{\pi}{4}\rho a^4, \qquad k = 2\left(\frac{8}{\pi^2}-\frac{1}{2}\right) = 0.621.$$

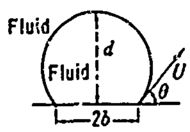
9. Fluid inside equilateral triangular prism rotating about its central axis:



$$T_1 = \frac{1}{80\sqrt{3}}\rho s^4\omega^2, \quad \text{as in Equation [103k],}$$

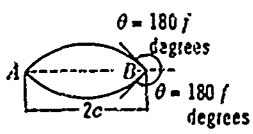
$$I_1' = \frac{1}{16\sqrt{3}}\rho s^4, \qquad k = \frac{2}{5}.$$

10. Lamina bent in form of circular arc, in translation at angle θ with chord:



$$T_1 = \frac{1}{2}\rho\pi\left(b^2\sin^2\theta + \frac{d^2}{2}\right)U^2, \quad \text{as in Equation [78r].}$$

11. Cylinder with contour consisting of two similar circular arcs; see Section 89.

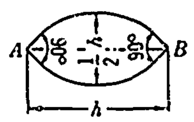


$$\text{Cross-sectional area } S = \frac{c^2}{\sin^2\theta}\left[2(1-f)\,\pi + \sin 2\theta\right]$$

1. Translation parallel to chord AB. $T_1 = \frac{1}{2}\rho k\, SU^2$, $\quad k = \frac{2\pi}{3}\left(\frac{1}{f^2}-1\right)\frac{c^2}{S} - 1.$

2. Translation perpendicular to chord AB: $T_1 = \frac{1}{2}\rho k\, SU^2$, $\quad k = \frac{2\pi}{3}\left(\frac{1}{2f^2}+1\right)\frac{c^2}{S} - 1.$

12. Cylinder with contour formed by two similar parabolic arcs meeting perpendicularly; see Section 91(d):



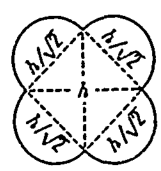
$$M_1' = \frac{1}{3}\rho h^2, \qquad T_1 = \frac{1}{2} k M_1' U^2;$$

1. Translation parallel to chord AB: $k = \dfrac{4K^4}{\pi^3} - 1 = 0.525.$

2. Translation perpendicular to chord AB: $k = \dfrac{8K^4}{\pi^3} - 1 = 2.049.$

Here $K = 1.8541$, the complete elliptic integral of modulus $\sqrt{1/2}$.

13. Cylinder whose contour is formed by four equal semicircles:

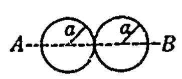


$M_1 = \dfrac{1}{4}(2+\pi)\,\rho h^2;$ for translation in any direction

$$T_1 = \frac{1}{2} k M_1 U^2, \qquad k = \frac{\pi}{2+\pi} K^2 - 1 = 1.100.$$

For K, see the preceding case. See Section 91(e).

14. Double circular cylinder, each cylinder of radius a; see Section 90:

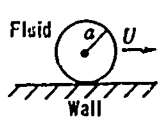


$$M_1' = 2\rho\pi a^2.$$

1. Translation parallel to line of axes AB: $T_1 = \rho\pi a^2 U^2\left(\dfrac{\pi^2}{6} - 1\right)$, $k = \dfrac{\pi^2}{6} - 1 = 0.645.$

2. Translation perpendicular to line of axes AB: $T_1 = \rho\pi a^2 U^2\left(\dfrac{\pi^2}{3} - 1\right)$, $k = \dfrac{\pi^2}{3} - 1 = 2.290.$

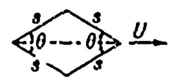
15. Cylinder of radius a sliding along fixed plane wall; see Section 90.



$$T_1 = \frac{1}{2}\rho\pi a^2 U^2\left(\frac{\pi^2}{3} - 1\right).$$

$$M_1' = \rho\pi a^2, \qquad k = \frac{\pi^2}{3} - 1 = 2.290.$$

16. Cylinder of rhombic cross-section, in translation along a diagonal; see Section 91(c).

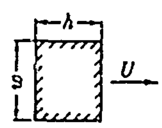


$$M_1' = \rho\, s^2 \sin\theta$$

$$T_1 = \frac{1}{2} k M_1 U^2, \qquad k = \frac{2\theta}{\sin\theta}\,\frac{\Gamma(3/2)}{\Gamma\left(1-\frac{\theta}{2\pi}\right)\Gamma\left(\frac{1}{2}+\frac{\theta}{2\pi}\right)} - 1.$$

Here θ is in radians and Γ stands for the gamma function.

17. Rectangular cylinder in translation parallel to a side; see Section 91(b) for references.

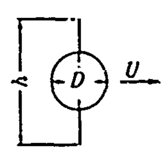


$M_1' = kM_1$ = apparent increase in mass,

$M_{10}'' = \rho\pi w^2/4$ or M_1'' for a plane lamina of width w.

$h/w = 0$	0.025	0.111	0.298	0.676	1.478	3.555	9.007	40.03
$M_1''/M_{10}'' = 1$	1.05	1.16	1.29	1.42	1.65	2.00	2.50	3.50

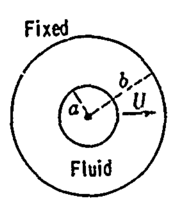
18. Circular cylinder with symmetrical fins:



$$T_1 = \frac{1}{2} k M_1 U^2, \qquad \text{as in Equation [91g],}$$

$$M_1' = \frac{1}{4}\rho\pi D^2, \qquad k = 1 + \left(\frac{h}{D} - \frac{D}{h}\right)^2.$$

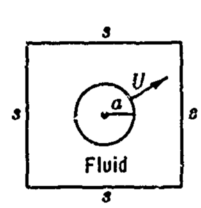
19. Cylinder of radius a in translation and instantaneously coaxial with enclosing fixed cylinder of radius b:



$$T_1 = \frac{1}{2}\rho\pi a^2 U^2 \frac{b^2+a^2}{b^2-a^2}, \qquad \text{as in Equation [104f]}$$

$$M_1' = \rho\pi a^2, \qquad k = \frac{b^2+a^2}{b^2-a^2}.$$

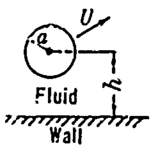
20. Cylinder of radius a in translation in any direction across axis of enclosing fixed square cylinder of side s, a/s small; see Section 91(l).



$$T_1 = \frac{1}{2}\rho\pi a^2 U^2 \left(1 + 6.38\frac{a^2}{s^2} \ldots\right),$$

$$M_1' = \rho\pi a^2, \qquad k = 1 + 6.88\frac{a^2}{s^2} \ldots$$

21. Cylinder of radius a in translation in any direction near a fixed infinite wall, a/h small:

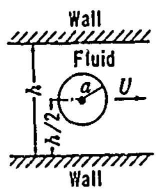


$$T_1 = \frac{1}{2}\rho\pi a^2 U^2 \left(1 + \frac{a^2}{2h^2} \dots\right), \quad \text{as in Equation [95g]}$$

$$M_1' = \rho\pi\, a^2, \qquad k = 1 + \frac{a^2}{2h^2} + \dots$$

(Only the force required to accelerate the cylinder is considered here.)

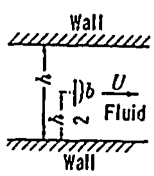
22. Cylinder of radius a moving symmetrically between fixed infinite walls h apart, a/h rather small:



$$T_1 = \frac{1}{2}\rho\pi a^2 U^2 \left[1 + \frac{2}{3}\left(\frac{\pi a}{h}\right)^2 + \dots\right], \quad \text{as in Equation [46q]}$$

$$M_1 = \rho\pi\, a^2, \qquad k = 1 + \frac{2}{3}\,\frac{\pi^2 a^2}{h^2} \dots\dots$$

23. Plane lamina of width b moving symmetrically between fixed infinite rigid walls h apart, b/h rather small:



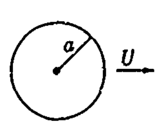
$$T_1 = \frac{1}{2}\rho\, U^2\, \frac{\pi b^2}{4}\left(1 + \frac{\pi^2 b^2}{24h^2} \dots\dots\right), \quad \text{as in Equation [65l]}$$

For the general case, see Section 65.

24. For kinetic energy around a Rankine cylinder, see Section 54.

B. THREE-DIMENSIONAL CASES

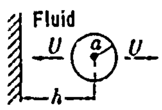
25. Sphere in translatory motion



$$T = \frac{\pi}{3}\rho\, a^3 U^2, \quad \text{as in Equation [127f]},$$

$$M' = \frac{4}{3}\pi\rho\, a^3, \qquad k = \frac{1}{2}.$$

26. Sphere moving perpendicularly to infinite rigid plane boundary, a/h small:

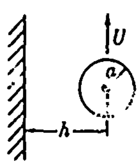


$$T = \frac{\pi}{3} \rho a^3 \left(1 + \frac{3}{8} \frac{a^3}{h^3} + \ldots\right) U^2,$$ as in Equation [130a] with $\alpha = 0$,

$$M' = \frac{4}{3} \pi \rho a^3, \qquad k = \frac{1}{2} \left(1 + \frac{3}{8} \frac{a^3}{h^3} + \ldots\right).$$

Only the force required to accelerate the sphere is considered here; see Section 130.

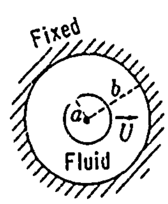
27. Sphere moving parallel to infinite rigid plane boundary, a/h small:



$$T = \frac{\pi}{3} \rho a^3 \left(1 + \frac{3}{16} \frac{a^3}{h^3} + \ldots\right) U^2,$$ as in Equation [130a] with $\alpha = 90$ deg,

$$M' = \frac{4}{3} \pi \rho a^3, \qquad k = \frac{1}{2} \left(1 + \frac{3}{16} \frac{a^3}{h^3} + \ldots\right).$$

28. Sphere moving past center of fixed spherical shell:



$$T = \frac{\pi}{3} \rho a^3 \frac{b^3 + 2a^3}{b^3 - a^3} U^2,$$ as in Equation [129e],

$$M' = \frac{4}{3} \pi \rho a^3, \qquad k = \frac{1}{2} \frac{b^3 + 2a^3}{b^3 - a^3}.$$

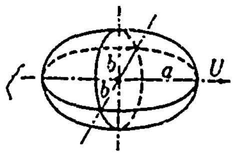
29. Prolate spheroid (or ovary ellipsoid), $a>b$; see Section 137:

Let e = eccentricity of sections through axis of symmetry,

$$\alpha_0 = \frac{1-e^2}{e^3} \left(\ln \frac{1+e}{1-e} - 2e\right),$$

$$\beta_0 = \frac{1-e^2}{e^3} \left(\frac{e}{1-e^2} - \frac{1}{2} \ln \frac{1+e}{1-e}\right).$$

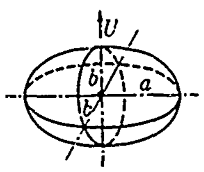
(1) Translation "end on":



$$T = \frac{2}{3} \rho\pi\, ab^2 U^2 \frac{\alpha_0}{2-\alpha_0},$$

$$M = \frac{4}{3} \rho\pi\, ab^2, \qquad k = k_1 = \frac{\alpha_0}{2-\alpha_0}.$$

(2) Translation "broadside on":

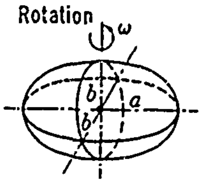


$$T = \frac{2}{3} \rho\pi\, ab^2 U^2 \frac{\beta_0}{2-\beta_0},$$

$$M = \frac{4}{3} \rho\pi\, ab^2, \qquad k = k_2 = \frac{\beta_0}{2-\beta_0}.$$

(3) Rotation about an axis perpendicular to axis of symmetry:

Axis of Rotation



$$T = \frac{1}{2} k' I \omega^2, \qquad I = \frac{4}{15} \rho\pi\, ab^2 (a^2+b^2),$$

$$k = k' = \frac{(a^2-b^2)^2 (\beta_0-\alpha_0)}{(a^2+b^2)\,[2(a^2-\beta^2) - (a^2+b^2)(\beta_0-\alpha_0)]}.$$

See Table 1, taken from Reference (1).

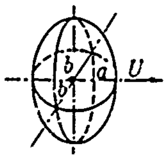
30. Oblate spheroid (or planetary ellipsoid), $a < b$, see Section 138, where $b = c$:

Let e = eccentricity of sections through axis of symmetry,

$$\alpha_0 = \frac{2}{e^3}(e - \sqrt{1-e^2}\ \sin^{-1} e),$$

$$\beta_0 = \frac{1}{e^3}[\ \sqrt{1-e^2}\ \sin^{-1} e\ - e\,(1-e^2)].$$

(1) Translation "broadside on" or parallel to axis:



$$T = \frac{2}{3} \rho\pi\, ab^2 U^2 \frac{\alpha_0}{2-\alpha_0},$$

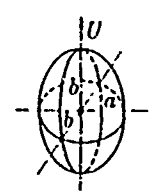
$$M = \frac{4}{3} \rho\pi\, ab^2, \qquad k = k_1 = \frac{\alpha_0}{2-\alpha_0}.$$

TABLE I

Coefficients of Inertia for Prolate Spheroid

a/b	k_1 Translation "end on"	k_2 Translation "broadside on"	k' Rotation about Minor Axis
1.00	0.500	0.500	0
1.50	0.305	0.621	0.094
2.00	0.209	0.702	0.240
2.51	0.156	0.763	0.367
2.99	0.122	0.803	0.465
3.99	0.082	0.860	0.608
4.99	0.059	0.895	0.701
6.01	0.045	0.918	0.764
6.97	0.036	0.933	0.805
8.01	0.029	0.945	0.840
9.02	0.024	0.954	0.865
9.97	0.021	0.960	0.883
∞	0	1.000	1.000

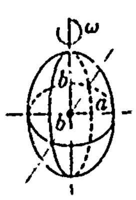
(2) Translation "edge on" or perpendicular to axis:



$$T = \frac{2}{3} \rho \pi a b^2 U^2 \frac{\beta_0}{2-\beta_0},$$

$$M = \frac{4}{3} \rho \pi a b^2, \qquad k = k_2 = \frac{\beta_0}{2-\beta_0}.$$

(3) Rotation about axis perpendicular to axis of symmetry:



$$T = \frac{1}{2} k' I \omega^2, \qquad I = \frac{4}{15} \rho \pi a b^2 (a^2 + b^2),$$

$$k = k' = \frac{(b^2-a^2)^2 (\alpha_0 - \beta_0)}{(a^2+b^2)\left[2(b^2-a^2) - (a^2+b^2)(\alpha_0-\beta_0)\right]}.$$

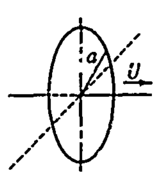
See Table II, in which k_1 and k_2 are from Reference (102).

TABLE II

Coefficients of Inertia for Oblate Spheroid

b/a	k_2 Translation "edge on"	k_1 Translation "broadside on"	k' Rotation about Equatorial Axis
1.00	0.500	0.500	0
1.50	0.384	0.803	0.115
2.00	0.310	1.118	0.337
2.50	0.260	1.428	0.587
3.00	0.223	1.742	0.840
4.00	0.174	2.379	1.330
5.00	0.140	3.000	1.978
6.00	0.121	3.642	2.259
7.00	0.105	4.279	2.697
8.00	0.092	4.915	3.150
9.00	0.084	5.549	3.697
10.00	0.075	6.183	4.019
∞	0.000	∞	∞

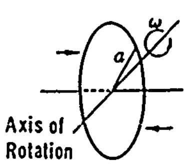
31. Circular disk in translation perpendicular to its faces:



$$T = \frac{4}{3}\rho a^3 U^2, \qquad \text{as in Equation [138o']};$$

$$\frac{\text{(apparent increase in mass)}}{\text{(spherical mass of fluid of radius } a)} = \frac{2}{\pi}.$$

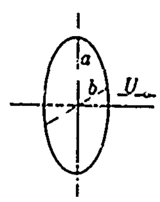
32. Circular disk rotating about a diameter; see Section 138:



$$T = \frac{8}{45}\rho a^5 \omega^2,$$

$$\frac{\text{(apparent increase in moment of inertia)}}{\text{(moment of inertia of sphere of fluid of radius } a \text{ or } 8\rho\pi a^5/15)} = \frac{2}{3}.$$

33. Elliptic disk of ellipticity e in translation perpendicular to its faces, $a > b$; References (240) and (235):



$$T = \frac{2\pi}{3E}\,\rho\, a^2 b\, U^2, \qquad e = \frac{1}{a}\sqrt{a^2 - b^2};$$

$$\frac{\text{(apparent increase in mass)}}{(\frac{4}{3}\rho\pi\, a^2 b = \text{ellipsoidal mass of fluid with axes } a,\ a,\ b)} = k'' = \frac{1}{E},$$

$$E = \int_0^{\pi/2} \sqrt{1 - e^2 \sin^2\theta}\; d\theta,$$ the complete elliptic integral of the second kind to modulus e; for table, see Peirce (20).

$a/b = 1$	1.25	1.5	1.75	2	2.5	3	4	6	9
$k'' = 0.637$	0.705	0.756	0.795	0.826	0.869	0.898	0.932	0.964	0.981

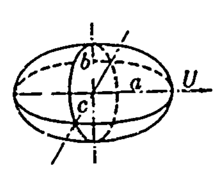
34. Ellipsoid, any ratio of the axes a, b, c; see Section 141:

$$\text{Let}\quad \alpha_0 = abc\int_0^\infty \frac{d\lambda}{(a^2+\lambda)^{3/2}\,(b^2+\lambda)^{1/2}\,(c^2+\lambda)^{1/2}},$$

$$\beta_0 = abc\int_0^\infty \frac{d\lambda}{(a^2+\lambda)^{1/2}\,(b^2+\lambda)^{3/2}\,(c^2+\lambda)^{1/2}},$$

$$\gamma_0 = abc\int_0^\infty \frac{d\lambda}{(a^2+\lambda)^{1/2}\,(b^2+\lambda)^{1/2}\,(c^2+\lambda)^{3/2}}.$$

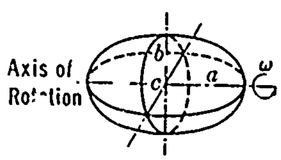
(1) Translation parallel to the a-axis:



$$T = \frac{2}{3}\rho\pi\, abc\, \frac{\alpha_0}{2-\alpha_0} U^2,$$

$$M' = \frac{4}{3}\rho\pi\, abc, \qquad k = \frac{\alpha_0}{2-\alpha_0}$$

(2) Rotation about the a-axis:



$$T = \frac{2}{15}\rho\pi\, abc\, \omega^2 \frac{(b^2-c^2)^2\ (\gamma_0-\beta_0)}{2(b^2-c^2) + (b^2+c^2)\ (\beta_0-\gamma_0)}$$

$$I' = \frac{4}{15}\rho\pi\, abc(b^2+c^2), \qquad k' = \frac{(b^2-c^2)^2\ (\gamma_0-\beta_0)}{2(b^4-c^4) + (b^2+c^2)^2\ (\beta_0-\gamma_0)}.$$

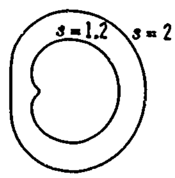
For the expression of α_0, β_0, γ_0 in terms of elliptic integrals, see N.A.C.A. Report 210 by Tuckerman (235) or Volume I of Durand's Aerodynamic Theory (3). Some values of k and of k', distinguished by a subscript to denote the axis of the motion, were given by Zahm (174).

35. Fluid inside ellipsoidal shell rotating about its a-axis, any relative magnitudes of a, b, c (see last figure):

$$T = \frac{2}{15}\rho\pi\, abc \frac{(b^2-c^2)^2}{b^2+c^2}\cdot \omega^2 \qquad \text{as in Equation [140f]},$$

$$I' = \frac{4}{15}\rho\pi\, abc\,(b^2+c^2), \qquad k = \left(\frac{b^2-c^2}{b^2+c^2}\right)^2.$$

36. Solid of revolution formed by revolving about its axis of symmetry the limason defined by $r = b\,(s + \cos\,\theta)/(s^2-1)$ where b and s are constants. The curve for $s = 1$ is a cardioid. A few values of k are given by Bateman in Reference (240):



$s = 1$	1.1	1.2	2	3	∞
$k = 0.578$	0.573	0.569	0.548	0.527	0.500.

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13. ABSTRACT

This book treats the known types of the potential flow of frictionless fluids having a uniform and invariable density. The main topics are: Chapter I, Fundamentals of the irrotational flow of frictionless fluids; Chapter II, The use of complex functions in hydrodynamics; Chapter III, Cases of two-dimensional flow – specific subjects treated are simple flow, line singularities, transformations, forces on cylinders, airfoil theory, rotating boundaries, flow in channels and free streamlines; Chapter IV, Cases of three-dimensional flow; Chapter V, Coefficients of inertia.

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